**Uitwerking extra opgaven hoofdstuk 6 Integraalrekening**

**6.1 Onbepaalde integralen**

Opgave 1

a $\int 5x^3\,\mathrm{d}x = \frac{5}{4}x^4 + C$

b $\int 8x^{10}\,\mathrm{d}x = \frac{8}{11}x^{11} + C$

c $\int \frac{7}{x^4}\,\mathrm{d}x = 7\int x^{-4}\,\mathrm{d}x = 7\cdot\frac{x^{-3}}{-3} + C = -\frac{7}{3x^3} + C$

d $\int \frac{4}{x}\,\mathrm{d}x = 4\ln|x| + C$

e $\int t^4x^5\,\mathrm{d}x = \frac{1}{6}t^4x^6 + C$

f $\int t^4x^5\,\mathrm{d}t = \frac{1}{5}t^5x^5 + C$

g $\int 9\sqrt{x}\,\mathrm{d}x = 9\int x^{\frac{1}{2}}\,\mathrm{d}x = 9\cdot\frac{2}{3}x^{\frac{3}{2}} + C = 6x\sqrt{x} + C$

h $\int \frac{2}{x\sqrt[3]{x}}\,\mathrm{d}x = 2\int x^{-\frac{4}{3}}\,\mathrm{d}x = 2\cdot(-3)\,x^{-\frac{1}{3}} + C = -\frac{6}{\sqrt[3]{x}} + C$

i $\int p^2\sqrt{p}\,\mathrm{d}p = \int p^{\frac{5}{2}}\,\mathrm{d}p = \frac{2}{7}p^2\sqrt{p} + C$

j $\int 5z\sqrt[5]{z^3}\,\mathrm{d}z = 5\int z^{\frac{8}{5}}\,\mathrm{d}z = 5\cdot\frac{5}{13}z^{\frac{13}{5}} + C = \frac{25}{13}z^2\sqrt[5]{z^3} + C$

k $\int 4\sin t\,\mathrm{d}t = -4\cos t + C$

l $\int \frac{4}{9+x^2}\,\mathrm{d}x = 4\int \frac{1}{3^2+x^2}\,\mathrm{d}x = \frac{4}{3}\arctan\left(\frac{x}{3}\right) + C$

Opgave 2

a

$$\int (6\cos x - 4\sin x)\,\mathrm{d}x = 6\int \cos x\,\mathrm{d}x - 4\int \sin x\,\mathrm{d}x = 6\sin x + 4\cos x + C$$

b

$$\int (8\cdot\mathrm{e}^x - 5\cdot 2^x)\,\mathrm{d}x = 8\int \mathrm{e}^x\,\mathrm{d}x - 5\int 2^x\,\mathrm{d}x = 8\cdot\mathrm{e}^x - \frac{5}{\ln 2}2^x + C$$

c

$$\int \left(5\sqrt{x}-3x\right)x^2\,\mathrm{d}x=\int \left(5x^2\sqrt{x}-3x^3\right)\mathrm{d}x$$
$$=5\int x^{\frac{5}{2}}\,\mathrm{d}x-3\int x^3\,\mathrm{d}x$$
$$=5\cdot\tfrac{2}{7}\cdot x^{\frac{7}{2}}-\tfrac{3}{4}x^4+C=\tfrac{10}{7}\cdot x^3\sqrt{x}-\tfrac{3}{4}x^4+C$$

d

$$\int \frac{7}{p^2+10}\,\mathrm{d}p=7\int \frac{1}{p^2+\left(\sqrt{10}\right)^2}\,\mathrm{d}p$$
$$=\frac{7}{\sqrt{10}}\arctan\left(\frac{p}{\sqrt{10}}\right)+C$$

e

$$\int \left(\frac{11}{x^2}-\frac{8}{x}+\sqrt[3]{x^2}\right)\mathrm{d}x=11\int x^{-2}\,\mathrm{d}x-8\int \frac{1}{x}\,\mathrm{d}x+\int x^{\frac{2}{3}}\,\mathrm{d}x$$
$$=-\frac{11}{x}-8\ln|x|+\tfrac{3}{5}x\sqrt[3]{x^2}+C$$

f

$$\int \left(x^2-4\right)^2x^3\,\mathrm{d}x=\int \left(x^4-8x^2+16\right)x^3\,\mathrm{d}x$$
$$=\int \left(x^7-8x^5+16x^3\right)\mathrm{d}x$$
$$=\tfrac{1}{8}x^8-\tfrac{4}{3}x^6+4x^4+C$$

g

$$\int \frac{(x+5)^2}{x}\,\mathrm{d}x=\int \frac{x^2+10x+25}{x}\,\mathrm{d}x=\int \frac{1}{x}\left(x^2+10x+25\right)\mathrm{d}x$$
$$=\int \left(x+10+\frac{25}{x}\right)\mathrm{d}x$$
$$=\tfrac{1}{2}x^2+10x+25\ln|x|+C$$

h

$$\int \frac{\sqrt[4]{x}+6x}{x^2}\,\mathrm{d}x = \int x^{-2}\left(x^{\frac{1}{4}}+6x\right)\mathrm{d}x$$

$$= \int \left(x^{-\frac{7}{4}}+\frac{6}{x}\right)\mathrm{d}x$$

$$= -\tfrac{4}{3}x^{-\frac{3}{4}}+6\ln|x|+C$$

$$= -\frac{4}{3\sqrt[4]{x^3}}+6\ln|x|+C$$

i $$\int \left(\tfrac{1}{3}\cos x+\sin x-\frac{7}{\cos^2 x}\right)\mathrm{d}x = \tfrac{1}{3}\sin x-\cos x-7\tan x+C$$

j $$\int \left(2\cdot \mathrm{e}^x-3\cdot \mathrm{e}^5+5^x+x^5\right)\mathrm{d}x = 2\cdot \mathrm{e}^x-3x\cdot \mathrm{e}^5+\frac{5^x}{\ln 5}+\tfrac{1}{6}x^6+C$$

k

$$\int \frac{\sqrt[5]{x^2}\cdot x\sqrt{x}}{\sqrt[3]{x}}\,\mathrm{d}x = \int \frac{x^{\frac{2}{5}}\cdot x^{\frac{3}{2}}}{x^{\frac{1}{3}}}\,\mathrm{d}x = \int x^{\frac{2}{5}+\frac{3}{2}-\frac{1}{3}}\,\mathrm{d}x$$

$$= \int x^{\frac{12+45-10}{30}}\,\mathrm{d}x = \int x^{\frac{47}{30}}\,\mathrm{d}x$$

$$= \frac{30}{77}x^{\frac{77}{30}}+C$$

$$= \frac{30}{77}x^2\cdot\sqrt[30]{x^{17}}+C$$

l

$$\int \left(\frac{t^3-6t}{t^5}+3\cdot 7^t\right)\mathrm{d}t = \int \left(t^{-2}-6t^{-4}+3\cdot 7^t\right)\mathrm{d}t$$

$$= -\frac{1}{t}+\frac{2}{t^3}+\frac{3\cdot 7^t}{\ln 7}+C$$

Opgave 3

$$\int f(x)\,\mathrm{d}x = \int \left(4x^3+\frac{5}{x}-3^x\right)\mathrm{d}x = x^4+5\ln|x|-\frac{3^x}{\ln 3}+C$$

Noem $F(x)=x^4+5\ln|x|-\dfrac{3^x}{\ln 3}+C$.

Berekenen van de constante *C*:

$$F(1)=5:\ 1+5\ln 1-\frac{3}{\ln 3}+C=5 \Rightarrow C=4+\frac{3}{\ln 3}$$

De gevraagde primitieve:

$$F(x)=x^4+5\ln|x|-\frac{3^x}{\ln 3}+4+\frac{3}{\ln 3}$$

Opgave 4

We berekenen achtereenvolgens de snelheid $v(t)$, het tijdstip waarop het projectiel begint te dalen, de hoogtefunctie $h(t)$ en uiteindelijk de bereikte hoogte.

$$\begin{aligned} v(t) &= \int a(t)\,\mathrm{d}t \\ &= -\int 9{,}81\,\mathrm{d}t \\ &= -9{,}81t + C_v \end{aligned}$$

Als $t = 0$ s, dan is $v = 100$ m/s, dit levert op de vergelijking $100 = 0 + C_v$, zodat $C_v = 100$ en $v(t) = -9{,}81t + 100$ (m/s).

Het projectiel begint te dalen op het moment dat $v(t) = 0$:

$$v(t) = 0 \Rightarrow -9{,}81t + 100 = 0 \Rightarrow t = \frac{100}{9{,}81} = 10{,}1936799 \text{ sec}$$

De hoogtefunctie:

$$\begin{aligned} h &= \int v\,\mathrm{d}t = \int (-9{,}81t + 100)\,\mathrm{d}t \\ &= -4{,}905t^2 + 100t + C_h \end{aligned}$$

Als $t = 0$ s, dan is $h = 0$ m, zodat $C_h = 0$ en $h(t) = -4{,}905t^2 + 100t$ (m).

De bereikte hoogte:

$$h(10{,}1936799) = 509{,}684 \text{ m}$$

Opgave 5

We berekenen achtereenvolgens de snelheid $v(t)$, de remtijd, de afstandsfunctie $s(t)$ en uiteindelijk de remweg.

$$\begin{aligned} v(t) &= \int a(t)\,\mathrm{d}t \\ &= -\int 4\sqrt{t}\,\mathrm{d}t \\ &= -4\int t^{\frac{1}{2}}\,\mathrm{d}t \\ &= -\tfrac{8}{3}t\sqrt{t} + C_v \end{aligned}$$

Als $t = 0$ s, dan is $v = \frac{120}{3{,}6}$ m/s, dit levert op de vergelijking $\frac{120}{3{,}6} = 0 + C_v$, zodat $C_v = \frac{120}{3{,}6}$ en $v(t) = -\frac{8}{3}t\sqrt{t} + \frac{120}{3{,}6}$ (m/s).

Vervolgens lossen we op: $v(t) = 0$.

$$v(t) = 0 \Rightarrow -\tfrac{8}{3}t\sqrt{t} + \tfrac{120}{3{,}6} = 0 \Rightarrow t^{\frac{3}{2}} = \frac{120\cdot 3}{3{,}6\cdot 8} \Rightarrow t = \left(\frac{120\cdot 3}{3{,}6\cdot 8}\right)^{\frac{2}{3}} = 5{,}3860867 \text{ sec}$$

De afstandsfunctie:

$$\begin{aligned} s &= \int v\,\mathrm{d}t = \int\left(-\tfrac{8}{3}t\sqrt{t} + \tfrac{120}{3{,}6}\right)\mathrm{d}t \\ &= -\tfrac{8}{3}\int t^{\frac{3}{2}}\,\mathrm{d}t + \tfrac{120}{3{,}6}\int 1\,\mathrm{d}t \\ &= -\tfrac{16}{15}t^2\sqrt{t} + \tfrac{120}{3{,}6}t + C_s \end{aligned}$$

Als $t=0$ s, dan is $s=0$ m, zodat $C_s=0$ en $s(t)=-\frac{16}{15}t^2\sqrt{t}+\frac{120}{3{,}6}t\ (\text{m})$.

De remweg:

$$s(5{,}3860867)=107{,}722\text{ m}$$

## 6.2 Bepaalde integralen

Opgave 1

a $$\int_0^{\frac{1}{2}\pi}\sin t\,\mathrm{d}t=[-\cos t]_0^{\frac{1}{2}\pi}=-\cos\left(\tfrac{1}{2}\pi\right)+\cos 0=1$$

b $$\int_1^3\frac{5}{x}\mathrm{d}x=\left[5\ln|x|\right]_1^3=5\ln 3$$

c $$\int_0^{\frac{1}{4}\pi}\frac{3}{\cos^2 x}\mathrm{d}x=[3\tan x]_0^{\frac{1}{4}\pi}=3\tan\left(\tfrac{1}{4}\pi\right)-3\tan 0=3$$

d $$\int_1^4\frac{1}{t\sqrt{t}}\mathrm{d}t=\int_1^4 t^{-\frac{3}{2}}\mathrm{d}t=\left[-2t^{-\frac{1}{2}}\right]_1^4=-\frac{2}{\sqrt{4}}+\frac{2}{\sqrt{1}}=-1+2=1$$

e $$\int_0^4\frac{2}{x^2+16}\mathrm{d}x=\frac{2}{4}\left[\arctan\left(\frac{x}{4}\right)\right]_0^4=\frac{1}{2}(\arctan 1-\arctan 0)=\frac{1}{8}\pi$$

f $$\int_0^{\frac{1}{4}\pi}\cos t\,\mathrm{d}t=[\sin t]_0^{\frac{1}{4}\pi}=\sin\left(\tfrac{1}{4}\pi\right)-\sin 0=\tfrac{1}{2}\sqrt{2}$$

g $$\int_0^1\left(p^2-p^3\right)\mathrm{d}p=\left[\tfrac{1}{3}p^3-\tfrac{1}{4}p^4\right]_0^1=\tfrac{1}{3}-\tfrac{1}{4}=\tfrac{1}{12}$$

h

$$\int_1^9\left(\frac{2}{\sqrt{x}}+3\sqrt{x}\right)\mathrm{d}x=\int_1^9\left(2x^{-\frac{1}{2}}+3x^{\frac{1}{2}}\right)\mathrm{d}x$$

$$=\left[4x^{\frac{1}{2}}+2x^{\frac{3}{2}}\right]_1^9$$

$$=4\cdot 3+2\cdot 9\cdot 3-(4+2)=60$$

i

$$\int_0^2\left(3\cdot\mathrm{e}^t-4^t+7\right)\mathrm{d}t=\left[3\cdot\mathrm{e}^t-\frac{4^t}{\ln 4}+7t\right]_0^2$$

$$=3\cdot\mathrm{e}^2-\frac{16}{\ln 4}+14-\left(3-\frac{1}{\ln 4}\right)=3\cdot\mathrm{e}^2-\frac{15}{\ln 4}+11$$

j

$$\int_0^5\left(5+x^2\sqrt{x}\right)\mathrm{d}x=\int_0^5\left(5+x^{\frac{5}{2}}\right)\mathrm{d}x$$

$$=\left[5x+\tfrac{2}{7}x^{\frac{7}{2}}\right]_0^5$$

$$=25+\tfrac{2}{7}\cdot125\sqrt{5}=25+\tfrac{250}{7}\sqrt{5}$$

k

$$\int_{-1}^{1}\left(3^x-x^3\right)\mathrm{d}x=\left[\frac{3^x}{\ln 3}-\tfrac{1}{4}x^4\right]_{-1}^{1}$$

$$=\frac{3}{\ln 3}-\frac{1}{4}-\left(\frac{1}{3\ln 3}-\frac{1}{4}\right)=\frac{8}{3\ln 3}$$

l

$$\int_1^4\frac{3-2s}{s\sqrt{s}}\,\mathrm{d}s=\int_1^4 s^{-\frac{3}{2}}\left(3-2s\right)\mathrm{d}s$$

$$=\int_1^4\left(3\cdot s^{-\frac{3}{2}}-2\cdot s^{-\frac{1}{2}}\right)\mathrm{d}s=\left[-\frac{6}{\sqrt{s}}-4\sqrt{s}\right]_1^4=-\tfrac{6}{2}-4\cdot2-\left(-6-4\right)=-1$$

Opgave 2

a $\quad O=\int_0^3 3^x\mathrm{d}x=\left[\frac{3^x}{\ln 3}\right]_0^3=\frac{27}{\ln 3}-\frac{1}{\ln 3}=\frac{26}{\ln 3}$

b $\quad O=\int_{-1}^4 3^x\mathrm{d}x=\left[\frac{3^x}{\ln 3}\right]_{-1}^4=\frac{81}{\ln 3}-\frac{1}{3\ln 3}=\frac{242}{3\ln 3}$

## 6.3 Integralen door substitutie

Opgave 1

a Stel $p=5x-3$, hieruit volgt $\frac{\mathrm{d}p}{\mathrm{d}x}=5$, zodat $\mathrm{d}x=\frac{1}{5}\mathrm{d}p$

Er volgt:

$$\int\left(5x-3\right)^8\mathrm{d}x=\tfrac{1}{5}\int p^8\,\mathrm{d}p$$

$$=\tfrac{1}{45}p^{10}+C=\tfrac{1}{45}\left(5x-3\right)^{10}+C$$

Of :

$$\int\left(5x-3\right)^8\mathrm{d}x=\tfrac{1}{5}\int\left(5x-3\right)^8\left(5x-8\right)'\,\mathrm{d}x$$

$$=\tfrac{1}{5}\int\left(5x-3\right)^8\mathrm{d}\left(5x-8\right)$$

$$=\tfrac{1}{45}\left(5x-3\right)^{10}+C$$

b

$$\int 5\cdot e^{8x}\,dx = \tfrac{5}{8}\int e^{8x}\cdot(8x)'\,dx$$
$$= \tfrac{5}{8}\int e^{8x}\,d(8x) = \tfrac{5}{8}e^{8x} + C$$

c

$$\int x^3\sqrt{2+5\cdot x^4}\,dx = \tfrac{1}{20}\int \left(2+5\cdot x^4\right)^{\frac{1}{2}}\left(2+5\cdot x^4\right)'\,dx$$
$$= \tfrac{1}{20}\int \left(2+5\cdot x^4\right)^{\frac{1}{2}}\,d\left(2+5\cdot x^4\right)$$
$$= \tfrac{1}{30}\left(2+5\cdot x^4\right)\sqrt{2+5\cdot x^4} + C$$

d

$$\int \frac{x}{\left(6-3x^2\right)^5}\,dx = -\tfrac{1}{6}\int p^{-5}\,dp$$
$$= -\tfrac{1}{6}\frac{p^{-4}}{-4} + C = \frac{1}{24\left(6-3x^2\right)^4} + C$$

e

Stel $p = 1+8t^3$, hieruit volgt $\frac{dp}{dt} = 24t^2$, zodat $t^2dt = \frac{1}{24}dp$

Grenzen: $t=0 \rightarrow p=1$ en $t=1 \rightarrow p=9$.

Uitvoeren van de substitutie levert op:

$$\int_0^1 5t^2\sqrt{1+8t^3}\,dt = \frac{5}{24}\int_1^9 p^{\frac{1}{2}}\,dp$$
$$= \frac{5}{24}\left[\frac{2}{3}p^{\frac{3}{2}}\right]_1^9 = \frac{5}{36}\left[p\sqrt{p}\,\right]_1^9$$
$$= \frac{5}{36}(9\cdot 3-1) = \frac{65}{18}$$

f

$$\int \frac{x^4}{1-x^5}\,dx = -\tfrac{1}{5}\int \frac{\left(1-x^5\right)'}{1-x^5}\,dx$$
$$= -\tfrac{1}{5}\int \frac{1}{1-x^5}\,d\left(1-x^5\right)$$
$$= -\tfrac{1}{5}\ln\left|1-x^5\right| + C$$

g

Stel $t = x^3-5$, hieruit volgt $\frac{dt}{dx} = 3x^2$, zodat $x^2dx = \frac{1}{3}dt$

Grenzen: $x=1 \rightarrow t=-4$ en $x=3 \rightarrow t=22$.

Uitvoeren van de substitutie levert op:

$$\int_1^3 2\cdot x^2\cdot e^{x^3-5}\,dx=\tfrac{2}{3}\int_{-4}^{22} e^t\,dt$$
$$=\tfrac{2}{3}\left[e^t\right]_{-4}^{22}=\tfrac{2}{3}\left(e^{22}-e^{-4}\right)$$

h

$$\int_0^{\frac{1}{3}\pi}\cos(x)\sin^3 x\,dx=\int_{x=0}^{\frac{1}{3}\pi}\sin^3 x\,d(\sin x)$$
$$=\left[\tfrac{1}{4}\sin^4 x\right]_0^{\frac{1}{3}\pi}=\tfrac{1}{4}\sin^4\left(\tfrac{1}{3}\pi\right)-0$$
$$=\tfrac{1}{4}\left(\tfrac{1}{2}\sqrt{3}\right)^4=\tfrac{1}{4}\cdot\tfrac{1}{16}\cdot 3^2=\frac{9}{64}$$

i Stel $t=5-2p^4$, hieruit volgt $\dfrac{dt}{dp}=-8p^3\Rightarrow p^3dp=-\tfrac{1}{8}dt$.

Uitvoeren van de substitutie levert op:

$$\int p^3\cdot 3^{5-2p^4}\,dp=-\tfrac{1}{8}\int 3^t\,dt=-\frac{1}{8}\frac{3^t}{\ln 3}+C=-\frac{3^{5-2p^4}}{8\ln 3}+C$$

j $\displaystyle\int\frac{1}{16+9x^2}dx=\int\frac{1}{16+(3x)^2}dx$

Stel $p=3x\Rightarrow\dfrac{dp}{dx}=3\Rightarrow dx=\dfrac{1}{3}dp$

Uitvoeren van de substitutie levert op:

$$\int\frac{1}{16+9x^2}dx=\int\frac{1}{16+(3x)^2}dx$$
$$=\tfrac{1}{3}\int\frac{1}{16+p^2}dp=\tfrac{1}{12}\arctan\left(\frac{p}{4}\right)+C$$
$$=\tfrac{1}{12}\arctan\left(\frac{3x}{4}\right)+C$$

Opgave 2

a

$$\int_0^4\frac{4}{x^2-6x+18}dx=4\int_0^4\frac{1}{(x-3)^2+9}dx$$
$$=4\int_{x=0}^4\frac{1}{(x-3)^2+9}d(x-3)=\tfrac{4}{3}\left[\arctan\left(\frac{x-3}{4}\right)\right]_0^4$$
$$=\tfrac{4}{3}\left(\arctan\tfrac{1}{4}-\arctan\left(-\tfrac{3}{4}\right)\right)=\tfrac{4}{3}\left(\arctan\tfrac{1}{4}+\arctan\left(\tfrac{3}{4}\right)\right)$$

b Stel $p=7-8t^4$, hieruit volgt $\dfrac{dp}{dt}=-32t^3\Rightarrow t^3dt=-\tfrac{1}{32}dp$.

Uitvoeren van de substitutie levert op:

$$\int \frac{4t^3}{\left(7-8t^4\right)^5}\,\mathrm{d}t = -\tfrac{1}{8}\int p^{-5}\,\mathrm{d}p$$

$$= -\tfrac{1}{8}\frac{p^{-4}}{-4}+C$$

$$= \frac{1}{32\left(7-8t^4\right)^4}+C$$

c

$$\int_{\mathrm{e}}^{5}\frac{1}{x\ln^4 x}\,\mathrm{d}x = \int_{\mathrm{e}}^{5}(\ln x)^{-4}\frac{1}{x}\,\mathrm{d}x = \int_{x=\mathrm{e}}^{5}(\ln x)^{-4}(\ln x)'\,\mathrm{d}x = \int_{\mathrm{e}}^{5}(\ln x)^{-4}\,\mathrm{d}(\ln x)$$

$$= \left[-\frac{1}{3(\ln x)^3}\right]_{\mathrm{e}}^{5} = -\frac{1}{3(\ln 5)^3}+\tfrac{1}{3}$$

d

$$\int_{\frac{1}{2}\pi}^{\pi}\cos^3 t\,\mathrm{d}t = \int_{\frac{1}{2}\pi}^{\pi}\cos^2 t\cdot\cos t\,\mathrm{d}t$$

$$= \int_{\frac{1}{2}\pi}^{\pi}\left(1-\sin^2 t\right)\mathrm{d}\sin t$$

$$= \left[\sin t-\tfrac{1}{3}\sin^3 t\right]_{\frac{1}{2}\pi}^{\pi} = 0-\left(1-\tfrac{1}{3}\right) = -\tfrac{2}{3}$$

e

$$\int\cos(3x)\cdot\sin(9x)\mathrm{d}x = \tfrac{1}{2}\int\left(\sin(12x)+\sin(6x)\right)\mathrm{d}x$$

$$= \tfrac{1}{2}\int\sin(12x)\mathrm{d}x+\tfrac{1}{2}\int\sin(6x)\mathrm{d}x$$

$$= \tfrac{1}{24}\int\sin(12x)\mathrm{d}(12x)+\tfrac{1}{12}\int\sin(6x)\mathrm{d}(6x)$$

$$= -\tfrac{1}{24}\cos(12x)-\tfrac{1}{12}\cos(6x)+C$$

f Stel $p=5-3x$. Uit $p=5-3x$ volgt $x=\tfrac{1}{3}(5-p)$ en $\mathrm{d}x=-\tfrac{1}{3}\mathrm{d}p$.

Uitvoeren van de substitutie levert op:

$$\int x\sqrt{5-3x}\,\mathrm{d}x = -\tfrac{1}{9}\int(5-p)\sqrt{p}\,\mathrm{d}p$$

$$= -\tfrac{1}{9}\int\left(5p^{\frac{1}{2}}-p^{\frac{3}{2}}\right)\mathrm{d}p = -\tfrac{1}{9}\left(5\cdot\tfrac{2}{3}p^{\frac{3}{2}}-\tfrac{2}{5}p^{\frac{5}{2}}\right)+C$$

$$= -\tfrac{10}{27}(5-3x)\sqrt{5-3x}+\tfrac{2}{45}(5-3x)^2\sqrt{5-3x}+C$$

g

$$\int_0^{\frac{1}{4}\pi} \sin^3 x \cos^2 x \,\mathrm{d}x = \int_0^{\frac{1}{4}\pi} \sin^2 x \cos^2 x \sin x \,\mathrm{d}x$$

$$= -\int_0^{\frac{1}{4}\pi} \left(1-\cos^2 x\right)\cos^2 x \,\mathrm{d}\left(\cos x\right)$$

$$= -\int_0^{\frac{1}{4}\pi} \left(\cos^2 x - \cos^4 x\right)\mathrm{d}\left(\cos x\right)$$

$$= -\left[\frac{1}{3}\cos^3 x - \frac{1}{5}\cos^5 x\right]_0^{\frac{1}{4}\pi}$$

$$= -\left(\frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3 - \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5 - \left(\frac{1}{3}-\frac{1}{5}\right)\right)$$

$$= -\frac{1}{6\sqrt{2}} + \frac{1}{20\sqrt{2}} + \frac{2}{15} = -\frac{7}{120}\sqrt{2} + \frac{2}{15} = \frac{16-7\sqrt{2}}{120}$$

h Stel $p = 1+4v$, hieruit volgt $\frac{\mathrm{d}p}{\mathrm{d}v} = 4$, zodat $\mathrm{d}v = \frac{1}{4}\mathrm{d}v$.

Uit $p = 1+4v$ volgt $v = \frac{1}{4}(p-1)$.

Grenzen: $v = 0 \to p = 1$ en $v = 2 \to p = 9$.

Uitvoeren van de substitutie levert op:

$$\int_0^2 \frac{3v}{\sqrt{1+4v}}\mathrm{d}v = \frac{3}{16}\int_1^9 \frac{p-1}{\sqrt{p}}\mathrm{d}p$$

$$= \frac{3}{16}\int_1^9 \left(p^{\frac{1}{2}} - p^{\frac{3}{2}}\right)\mathrm{d}p$$

$$= \frac{3}{16}\left[\frac{2}{3}p\sqrt{p} - 2\sqrt{p}\right]_1^9 = \frac{3}{16}\left(\frac{2}{3}\cdot 9\cdot 3 - 2\cdot 3\right) - \frac{3}{16}\left(\frac{2}{3} - 2\right) =$$

$$= \frac{3}{16}\cdot 12 - \frac{3}{16}\left(-\frac{4}{3}\right) = \frac{15}{6} = \frac{5}{2}$$

i $\int x^5\sqrt{1+x^2}\,\mathrm{d}x=\int x^4\cdot x\sqrt{1+x^2}\,\mathrm{d}x$

Stel $p=1+x^2$, hieruit volgt $\frac{\mathrm{d}p}{\mathrm{d}x}=2x\Rightarrow x\mathrm{d}x=\frac{1}{2}\mathrm{d}p$.

Uit $p=1+x^2$ volgt $x^2=p-1$.

Uitvoeren van de substitutie levert op:

$$\begin{aligned}\int x^5\sqrt{1+x^2}\,\mathrm{d}x&=\int x^4\cdot x\sqrt{1+x^2}\,\mathrm{d}x\\&=\tfrac{1}{2}\int (p-1)^2\sqrt{p}\,\mathrm{d}p=\tfrac{1}{2}\int \left(p^2-2p+1\right)\sqrt{p}\,\mathrm{d}p\\&=\tfrac{1}{2}\int\left(p^{\frac{5}{2}}-2p^{\frac{3}{2}}+p^{\frac{1}{2}}\right)\mathrm{d}p\\&=\tfrac{1}{2}\left(\tfrac{2}{7}p^{\frac{7}{2}}-2\cdot\tfrac{2}{5}p^{\frac{5}{2}}+\tfrac{2}{3}p^{\frac{3}{2}}\right)+C\\&=\tfrac{1}{7}\left(1+x^2\right)^3\sqrt{1+x^2}-\tfrac{2}{5}\left(1+x^2\right)^2\sqrt{1+x^2}+\tfrac{1}{3}\left(1+x^2\right)\sqrt{1+x^2}+C\end{aligned}$$

j Gebruik $\sin^2(x)=\frac{1}{2}(1-\cos(2x))$, zodat $\sin^2\left(y^5\right)=\frac{1}{2}\left(1-\cos\left(2y^5\right)\right)$

Er volgt:

$$\begin{aligned}\int y^4\sin^2\left(y^5\right)\mathrm{d}y&=\tfrac{1}{2}\int y^4\left(1-\cos\left(2y^5\right)\right)\mathrm{d}y\\&=\tfrac{1}{2}\int y^4\,\mathrm{d}y-\tfrac{1}{2}\int y^4\cos\left(2y^5\right)\mathrm{d}y\\&=\tfrac{1}{2}\int y^4\,\mathrm{d}y-\tfrac{1}{2}\cdot\tfrac{1}{10}\int\cos\left(2y^5\right)\mathrm{d}\left(2y^5\right)\\&=\tfrac{1}{10}y^5-\tfrac{1}{20}\sin\left(2y^5\right)+C\end{aligned}$$

## 6.4 Partiële integratie

Opgave 1

a

$$\begin{aligned}\int x^5\ln x\,\mathrm{d}x&=\int x^5\ln x\,\mathrm{d}\left(\tfrac{1}{6}x^6\right)\\&=\tfrac{1}{6}x^6\ln x-\tfrac{1}{6}\int x^6\,\mathrm{d}(\ln x)\\&=\tfrac{1}{6}x^6\ln x-\tfrac{1}{6}\int x^6\cdot\frac{1}{x}\mathrm{d}x\\&=\tfrac{1}{6}x^6\ln x-\tfrac{1}{6}\int x^5\mathrm{d}x\\&=\tfrac{1}{6}x^6\ln x-\tfrac{1}{36}x^6+C\end{aligned}$$

b We moeten de factor $\mathrm{e}^{3x}$ achter het d-teken brengen. Om $\mathrm{e}^{3x}$ achter het d-teken te kunnen brengen, bepalen we eerst $\int \mathrm{e}^{3x}\,\mathrm{d}x$ :

$\int \mathrm{e}^{3x}\,\mathrm{d}x = \frac{1}{3}\int \mathrm{e}^{3x}\,\mathrm{d}(3x) = \frac{1}{3}\mathrm{e}^{3x} + C$, zodat $\mathrm{e}^{3x}\mathrm{d}x = \mathrm{d}\left(\frac{1}{3}\mathrm{e}^{3x}\right)$.

Er volgt

$$\begin{aligned}\int_0^1 x\cdot \mathrm{e}^{3x}\,\mathrm{d}x &= \int_{x=0}^{1} x\,\mathrm{d}\left(\tfrac{1}{3}\mathrm{e}^{3x}\right)\\ &= \tfrac{1}{3}\left[x\cdot \mathrm{e}^{3x}\right]_0^1 - \tfrac{1}{3}\int_0^1 \mathrm{e}^{3x}\,\mathrm{d}x\\ &= \tfrac{1}{3}\mathrm{e}^3 - \tfrac{1}{9}\left[\mathrm{e}^{3x}\right]_0^1\\ &= \tfrac{1}{3}\mathrm{e}^3 - \tfrac{1}{9}\left(\mathrm{e}^3 - 1\right) = \tfrac{2}{9}\mathrm{e}^3 + \tfrac{1}{9}\end{aligned}$$

c

$$\begin{aligned}\int t^2\cdot 4^t\,\mathrm{d}t &= \int t^2\,\mathrm{d}\left(\frac{4^t}{\ln 4}\right)\\ &= \frac{4^t\cdot t^2}{\ln 4} - \frac{1}{\ln 4}\int 4^t\,\mathrm{d}t^2\\ &= \frac{4^t\cdot t^2}{\ln 4} - \frac{2}{\ln 4}\int 4^t\cdot t\,\mathrm{d}t\\ &= \frac{4^t\cdot t^2}{\ln 4} - \frac{2}{\ln 4}\int t\,\mathrm{d}\left(\frac{4^t}{\ln 4}\right)\\ &= \frac{4^t\cdot t^2}{\ln 4} - \frac{2}{\ln 4}\left(\frac{4^t\cdot t}{\ln 4} - \frac{1}{\ln 4}\int 4^t\,\mathrm{d}t\right)\\ &= \frac{4^t\cdot t^2}{\ln 4} - \frac{2t\cdot 4^t}{(\ln 4)^2} + \frac{2\cdot 4^t}{(\ln 4)^3} + C\end{aligned}$$

d Gebruik $\sin(2x)\mathrm{d}x = \mathrm{d}\left(-\frac{1}{2}\cos(2x)\right)$

Er volgt:

$$\begin{aligned}\int_0^{\pi} x\sin(2x)\mathrm{d}x &= \int_{x=0}^{\pi} x\mathrm{d}\left(-\tfrac{1}{2}\cos(2x)\right)\\ &= \left[-\tfrac{1}{2}x\cos(2x)\right]_0^{\pi} + \tfrac{1}{2}\int_0^{\pi}\cos(2x)\mathrm{d}x\\ &= -\tfrac{1}{2}\pi\cos(2\pi) + 0 + \tfrac{1}{4}\left[\sin(2x)\right]_0^{\pi}\\ &= -\tfrac{1}{2}\pi + \tfrac{1}{4}\left(\sin(2\pi) - \sin 0\right) = -\tfrac{1}{2}\pi\end{aligned}$$

e We voeren direct partiële integratie uit volgens de formule

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x\,,$$

waarbij $f(x) = 5\cdot\arctan 2x$ en $g(x) = x$:

$$\int 5\arctan(2x)\,\mathrm{d}x = 5x\arctan(2x) - 5\int x\,\mathrm{d}(\arctan(2x)) \quad \left(\mathrm{d}(\arctan 2x) = \frac{2}{1+4x^2}\mathrm{d}x\right)$$

$$= 5x\arctan(2x) - 10\int \frac{x}{1+4x^2}\mathrm{d}x$$

$$= 5x\arctan(2x) - \tfrac{10}{8}\int \frac{(1+4x^2)'}{1+4x^2}\mathrm{d}x$$

$$= 5x\arctan(2x) - \tfrac{5}{4}\int \frac{1}{1+4x^2}\mathrm{d}(1+4x^2)$$

$$= 5x\arctan(2x) - \tfrac{5}{4}\ln(1+4x^2) + C$$

f

$$\int_1^{\mathrm{e}} \sqrt{x}\ln(x)\mathrm{d}x = \int_1^{\mathrm{e}} \ln(x)\mathrm{d}\left(\tfrac{2}{3}x^{\frac{3}{2}}\right)$$

$$= \left[\tfrac{2}{3}x\sqrt{x}\ln(x)\right]_1^{\mathrm{e}} - \tfrac{2}{3}\int_1^{\mathrm{e}} x\sqrt{x}\,\mathrm{d}\ln x$$

$$= \tfrac{2}{3}\mathrm{e}\sqrt{\mathrm{e}} - \tfrac{2}{3}\int_1^{\mathrm{e}} \sqrt{x}\,\mathrm{d}x$$

$$= \tfrac{2}{3}\mathrm{e}\sqrt{\mathrm{e}} - \tfrac{2}{3}\left[\tfrac{2}{3}x\sqrt{x}\right]_1^{\mathrm{e}}$$

$$= \tfrac{2}{3}\mathrm{e}\sqrt{\mathrm{e}} - \tfrac{4}{9}\left(\mathrm{e}\sqrt{\mathrm{e}} - 1\right) = \tfrac{2}{9}\mathrm{e}\sqrt{\mathrm{e}} + \tfrac{4}{9}$$

g

$$\int 9t^2\cdot\mathrm{e}^{3t}\,\mathrm{d}t = 9\int t^2\,\mathrm{d}\left(\tfrac{1}{3}\mathrm{e}^{3t}\right)$$

$$= 3t^2\cdot\mathrm{e}^{3t} - 3\int \mathrm{e}^{3t}\,\mathrm{d}(t^2)$$

$$= 3t^2\cdot\mathrm{e}^{3t} - 6\int t\cdot\mathrm{e}^{3t}\,\mathrm{d}t$$

$$= 3t^2\cdot\mathrm{e}^{3t} - 6\int t\,\mathrm{d}\left(\tfrac{1}{3}\mathrm{e}^{3t}\right)$$

$$= 3t^2\cdot\mathrm{e}^{3t} - 2t\cdot\mathrm{e}^{3t} + 2\int \mathrm{e}^{3t}\,\mathrm{d}t$$

$$= 3t^2\cdot\mathrm{e}^{3t} - 2t\cdot\mathrm{e}^{3t} + \tfrac{2}{3}\mathrm{e}^{3t} + C$$

h

$$\int (5x-4)\cos(3x)\,\mathrm{d}x = \int (5x-4)\,\mathrm{d}\left(\tfrac{1}{3}\sin(3x)\right)$$
$$= \tfrac{1}{3}(5x-4)\sin(3x) - \tfrac{1}{3}\int \sin(3x)\,\mathrm{d}(5x-4)$$
$$= \tfrac{1}{3}(5x-4)\sin(3x) - \tfrac{5}{3}\int \sin(3x)\,\mathrm{d}x$$
$$= \tfrac{1}{3}(5x-4)\sin(3x) - \tfrac{5}{9}\int \sin(3x)\,\mathrm{d}(3x)$$
$$= \tfrac{1}{3}(5x-4)\sin(3x) + \tfrac{5}{9}\cos(3x) + C$$

**6.5 Integratie door middel van breuksplitsing**

Opgave 1

a De noemer is te ontbinden als $x^2 - 8x - 20 = (x+2)(x-10)$.

Breuksplitsen leidt tot: $\dfrac{4x}{(x+2)(x-10)} = \dfrac{A}{x-10} + \dfrac{B}{x+2}$. Hieruit volgt:

$$4x = A(x+2) + B(x-10)$$

Kies $x = 10$: $40 = 12A \Rightarrow A = \frac{10}{3}$

Kies $x = -2$: $-8 = -12B \Rightarrow B = \frac{2}{3}$

Er volgt:

$$\int \frac{4x}{x^2-8x-20}\,\mathrm{d}x = \tfrac{10}{3}\int \frac{1}{x-10}\,\mathrm{d}x + \tfrac{2}{3}\int \frac{1}{x+2}\,\mathrm{d}x$$
$$= \tfrac{10}{3}\int \frac{1}{x-10}\,\mathrm{d}(x-10) + \tfrac{2}{3}\int \frac{1}{x+2}\,\mathrm{d}(x+2)$$
$$= \tfrac{10}{3}\ln|x-10| + \tfrac{2}{3}\ln|x+2| + C$$

b De ontbinding van de noemer: $x^2 - 16 = (x-4)(x+4)$.

De breuksplitsen is: $\dfrac{3}{(x-4)(x+4)} = \dfrac{A}{x-4} + \dfrac{B}{x+4}$. Hieruit volgt:

$$3 = A(x+4) + B(x-4)$$

Kies $x = 4$: $3 = 8A \Rightarrow A = \frac{3}{8}$

Kies $x = -4$: $3 = -8B \Rightarrow B = -\frac{3}{8}$

Er volgt:

$$\int \frac{3}{x^2-16}\,\mathrm{d}x = \tfrac{3}{8}\int \frac{1}{x-4}\,\mathrm{d}x - \tfrac{3}{8}\int \frac{1}{x+4}\,\mathrm{d}x$$
$$= \tfrac{3}{8}\int \frac{1}{x-4}\,\mathrm{d}(x-4) - \tfrac{3}{8}\int \frac{1}{x+4}\,\mathrm{d}(x+4)$$
$$= \tfrac{3}{8}\ln|x-4| - \tfrac{3}{8}\ln|x+4| + C$$
$$= \tfrac{3}{8}\ln\left|\frac{x-4}{x+4}\right| + C$$

c De noemer is te ontbinden als $p^3-4p^2+4p = p(p-2)^2$.

Breuksplitsen leidt tot: $\dfrac{p+4}{p^3-4p^2+4p} = \dfrac{A}{p} + \dfrac{B}{p-2} + \dfrac{C}{(p-2)^2}$. Hieruit volgt:

$$p+4 = A(p-2)^2 + B\cdot p(p-2) + C\cdot p$$

Kies $p=0$: $4=4A \Rightarrow A=1$
Kies $p=2$: $6=2C \Rightarrow C=3$
Coëfficiënt van $p^2$: $0=A+B \Rightarrow B=-1$.
Er volgt:

$$\int \frac{p+4}{p^3-4p^2+4p}\,\mathrm{d}p = \int \frac{1}{p}\,\mathrm{d}p - \int \frac{1}{p-2}\,\mathrm{d}p + 3\int \frac{1}{(p-2)^2}\,\mathrm{d}p$$
$$= \int \frac{1}{p}\,\mathrm{d}p - \int \frac{1}{p-2}\,\mathrm{d}(p-2) + 3\int (p-2)^{-2}\,\mathrm{d}(p-2)$$
$$= \ln|p| - \ln|p-2| - \frac{3}{p-2} + C$$

d Breuksplitsen leidt tot: $\dfrac{x}{(x-3)^2} = \dfrac{A}{x-3} + \dfrac{B}{(x-3)^2}$. Hieruit volgt:

$$x = A(x-3) + B$$

Kies $x=3$: $3=B \Rightarrow B=3$
Coëfficiënt van $x$: $1=A \Rightarrow A=1$.
Er volgt:

$$\int_0^2 \frac{x}{x^2-6x+9}\,\mathrm{d}x = \int_0^2 \frac{1}{x-3}\,\mathrm{d}x + 3\int_0^2 \frac{1}{(x-3)^2}\,\mathrm{d}x$$
$$= \int_0^2 \frac{1}{x-3}\,\mathrm{d}(x-3) + 3\int_0^2 (x-3)^{-2}\,\mathrm{d}(x-3)$$
$$= \left[\ln|x-3|\right]_0^2 - 3\left[\frac{1}{x-3}\right]_0^2$$
$$= \ln|-1| - \ln|-3| - 3\left(\tfrac{1}{-1} - \tfrac{1}{-3}\right)$$
$$= -\ln(3) + 2$$

e De ontbinding van de noemer: $t^3-6t^2=t^2(t-6)$.

De breuksplitsen:

$$\frac{12}{t^2(t-6)}=\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t-6}.$$

Hieruit volgt:

$$12=At(t-6)+B(t-6)+Ct^2$$

Kies $t=0$: $12=-6B\Rightarrow B=-2$

Kies $t=6$: $12=36C\Rightarrow C=\frac{1}{3}$

Coëfficiënt van $t^2$: $0=A+C\Rightarrow A=-\frac{1}{3}$.

Er volgt:

$$\int\frac{12}{t^3-6t^2}\mathrm{d}t=-\tfrac{1}{3}\int\frac{1}{t}\mathrm{d}t-2\int\frac{1}{t^2}\mathrm{d}t+\tfrac{1}{3}\int\frac{1}{t-6}\mathrm{d}t$$

$$=-\tfrac{1}{3}\ln|t|+\frac{2}{t}+\tfrac{1}{3}\ln|t-6|+C$$

f

$$\int\frac{4x+5}{x^2+6x+13}\mathrm{d}x=2\int\frac{2x+6}{x^2+6x+13}\mathrm{d}x-7\int\frac{1}{x^2+6x+13}\mathrm{d}x$$

$$=2\int\frac{1}{x^2+6x+13}\mathrm{d}\left(x^2+6x+13\right)-7\int\frac{1}{(x+3)^2+4}\mathrm{d}(x+3)$$

$$=2\ln\left(x^2+2x+5\right)-\tfrac{7}{2}\arctan\left(\frac{x+3}{2}\right)+C$$

g

$$\int_1^2\frac{x}{x^2-8x+20}\mathrm{d}x=\tfrac{1}{2}\int_1^2\frac{2x-8}{x^2-8x+20}\mathrm{d}x+4\int_1^2\frac{1}{x^2-8x+20}\mathrm{d}x$$

$$=\tfrac{1}{2}\int_1^2\frac{1}{x^2-8x+20}\mathrm{d}\left(x^2-8x+20\right)+4\int_1^2\frac{1}{(x-4)^2+4}\mathrm{d}(x-4)$$

$$=\tfrac{1}{2}\left[\ln\left|x^2-8x+20\right|\right]_1^2+2\left[\arctan\left(\frac{x-4}{2}\right)\right]_1^2$$

$$=\tfrac{1}{2}(\ln 8-\ln 13)+2\left(\arctan(-1)-\arctan\left(-\tfrac{3}{2}\right)\right)$$

$$=\tfrac{1}{2}\ln\left(\tfrac{8}{13}\right)-\tfrac{1}{2}\pi+2\arctan\left(\tfrac{3}{2}\right)$$

h De ontbinding van de noemer: $x^4 - 7x^3 + 10x^2 = x^2(x-5)(x-2)$.

Breuksplitsen leidt tot: $\dfrac{x+10}{x^2(x-5)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-5} + \dfrac{D}{x-2}$. Hieruit volgt:

$$x + 10 = Ax(x-5)(x-2) + B(x-5)(x-2) + C \cdot x^2(x-2) + D \cdot x^2(x-5)$$

Kies $x = 0$: $10 = 10B \Rightarrow B = 1$

Kies $x = 2$: $12 = -12D \Rightarrow D = -1$

Kies $x = 5$: $15 = 75C \Rightarrow C = \frac{1}{5}$

Coëfficiënt van $x^3$: $0 = A + C + D \Rightarrow A = -C - D = -\frac{1}{5} + 1 = \frac{4}{5}$.

Er volgt:

$$\int \frac{x+10}{x^4 - 7x^3 + 10x^2}\,\mathrm{d}x = \frac{4}{5}\int \frac{1}{x}\,\mathrm{d}x + \int \frac{1}{x^2}\,\mathrm{d}x + \frac{1}{5}\int \frac{1}{x-5}\,\mathrm{d}x - \int \frac{1}{x-2}\,\mathrm{d}x$$

$$= \frac{4}{5}\ln|x| - \frac{1}{x} + \frac{1}{5}\ln|x-5| - \ln|x-2| + C$$