



Uitwerking extra opgaven hoofdstuk 6 Integraalrekening

6.1 Onbepaalde integralen

Opgave 1

a $\int 5x^3 dx = \frac{5}{4}x^4 + C$

b $\int 8x^{10} dx = \frac{8}{11}x^{11} + C$

c $\int \frac{7}{x^4} dx = 7 \int x^{-4} dx = 7 \cdot \frac{x^{-3}}{-3} + C = -\frac{7}{3x^3} + C$

d $\int \frac{4}{x} dx = 4 \ln|x| + C$

e $\int t^4 x^5 dx = \frac{1}{6}t^4 x^6 + C$

f $\int t^4 x^5 dt = \frac{1}{5}t^5 x^5 + C$

g $\int 9\sqrt{x} dx = 9 \int x^{\frac{1}{2}} dx = 9 \cdot \frac{2}{3}x^{\frac{3}{2}} + C = 6x\sqrt{x} + C$

h $\int \frac{2}{x\sqrt[3]{x}} dx = 2 \int x^{-\frac{4}{3}} dx = 2 \cdot (-3)x^{-\frac{1}{3}} + C = -\frac{6}{\sqrt[3]{x}} + C$

i $\int p^2 \sqrt{p} dp = \int p^{\frac{5}{2}} dp = \frac{2}{7}p^2 \sqrt{p} + C$

j $\int 5z \sqrt[5]{z^3} dz = 5 \int z^{\frac{8}{5}} dz = 5 \cdot \frac{5}{13}z^{\frac{13}{5}} + C = \frac{25}{13}z^2 \sqrt[5]{z^3} + C$

k $\int 4 \sin t dt = -4 \cos t + C$

l $\int \frac{4}{9+x^2} dx = 4 \int \frac{1}{3^2+x^2} dx = \frac{4}{3} \arctan\left(\frac{x}{3}\right) + C$

Opgave 2

a
$$\begin{aligned} \int (6 \cos x - 4 \sin x) dx &= 6 \int \cos x dx - 4 \int \sin x dx \\ &= 6 \sin x + 4 \cos x + C \end{aligned}$$

b
$$\begin{aligned} \int (8 \cdot e^x - 5 \cdot 2^x) dx &= 8 \int e^x dx - 5 \int 2^x dx \\ &= 8 \cdot e^x - \frac{5}{\ln 2} 2^x + C \end{aligned}$$



c

$$\begin{aligned}\int (5\sqrt{x} - 3x)x^2 dx &= \int (5x^2\sqrt{x} - 3x^3) dx \\ &= 5 \int x^{\frac{5}{2}} dx - 3 \int x^3 dx \\ &= 5 \cdot \frac{2}{7} \cdot x^{\frac{7}{2}} - \frac{3}{4} x^4 + C = \frac{10}{7} \cdot x^3 \sqrt{x} - \frac{3}{4} x^4 + C\end{aligned}$$

d

$$\begin{aligned}\int \frac{7}{p^2 + 10} dp &= 7 \int \frac{1}{p^2 + (\sqrt{10})^2} dp \\ &= \frac{7}{\sqrt{10}} \arctan\left(\frac{p}{\sqrt{10}}\right) + C\end{aligned}$$

e

$$\begin{aligned}\int \left(\frac{11}{x^2} - \frac{8}{x} + \sqrt[3]{x^2}\right) dx &= 11 \int x^{-2} dx - 8 \int \frac{1}{x} dx + \int x^{\frac{2}{3}} dx \\ &= -\frac{11}{x} - 8 \ln|x| + \frac{3}{5} x^{\frac{5}{3}} + C\end{aligned}$$

f

$$\begin{aligned}\int (x^2 - 4)^2 x^3 dx &= \int (x^4 - 8x^2 + 16)x^3 dx \\ &= \int (x^7 - 8x^5 + 16x^3) dx \\ &= \frac{1}{8} x^8 - \frac{4}{3} x^6 + 4x^4 + C\end{aligned}$$

g

$$\begin{aligned}\int \frac{(x+5)^2}{x} dx &= \int \frac{x^2 + 10x + 25}{x} dx = \int \frac{1}{x}(x^2 + 10x + 25) dx \\ &= \int \left(x + 10 + \frac{25}{x}\right) dx \\ &= \frac{1}{2} x^2 + 10x + 25 \ln|x| + C\end{aligned}$$



h

$$\begin{aligned}\int \frac{\sqrt[4]{x} + 6x}{x^2} dx &= \int x^{-2} \left(x^{\frac{1}{4}} + 6x \right) dx \\ &= \int \left(x^{-\frac{7}{4}} + \frac{6}{x} \right) dx \\ &= -\frac{4}{3} x^{-\frac{3}{4}} + 6 \ln|x| + C \\ &= -\frac{4}{3\sqrt[4]{x^3}} + 6 \ln|x| + C\end{aligned}$$

i

$$\int \left(\frac{1}{3} \cos x + \sin x - \frac{7}{\cos^2 x} \right) dx = \frac{1}{3} \sin x - \cos x - 7 \tan x + C$$

j

$$\int (2 \cdot e^x - 3 \cdot e^5 + 5^x + x^5) dx = 2 \cdot e^x - 3x \cdot e^5 + \frac{5^x}{\ln 5} + \frac{1}{6} x^6 + C$$

k

$$\begin{aligned}\int \frac{\sqrt[5]{x^2} \cdot x \sqrt{x}}{\sqrt[3]{x}} dx &= \int \frac{x^{\frac{2}{5}} \cdot x^{\frac{3}{2}}}{x^{\frac{1}{3}}} dx = \int x^{\frac{2}{5} + \frac{3}{2} - \frac{1}{3}} dx \\ &= \int x^{\frac{12+45-10}{30}} dx = \int x^{\frac{47}{30}} dx \\ &= \frac{30}{77} x^{\frac{77}{30}} + C \\ &= \frac{30}{77} x^2 \cdot \sqrt[30]{x^{17}} + C\end{aligned}$$

l

$$\begin{aligned}\int \left(\frac{t^3 - 6t}{t^5} + 3 \cdot 7^t \right) dt &= \int (t^{-2} - 6t^{-4} + 3 \cdot 7^t) dt \\ &= -\frac{1}{t} + \frac{2}{t^3} + \frac{3 \cdot 7^t}{\ln 7} + C\end{aligned}$$

Opgave 3

$$\int f(x) dx = \int \left(4x^3 + \frac{5}{x} - 3^x \right) dx = x^4 + 5 \ln|x| - \frac{3^x}{\ln 3} + C$$

Noem $F(x) = x^4 + 5 \ln|x| - \frac{3^x}{\ln 3} + C$.

Berekenen van de constante C:

$$F(1) = 5 : 1 + 5 \ln 1 - \frac{3}{\ln 3} + C = 5 \Rightarrow C = 4 + \frac{3}{\ln 3}$$

De gevraagde primitieve:

$$F(x) = x^4 + 5 \ln|x| - \frac{3^x}{\ln 3} + 4 + \frac{3}{\ln 3}$$



Toegepaste Wiskunde deel 1

Opgave 4

We berekenen achtereenvolgens de snelheid $v(t)$, het tijdstip waarop het projectiel begint te dalen, de hoogtefunctie $h(t)$ en uiteindelijk de bereikte hoogte.

$$\begin{aligned}v(t) &= \int a(t) dt \\ &= -\int 9,81 dt \\ &= -9,81t + C_v\end{aligned}$$

Als $t = 0$ s, dan is $v = 100$ m/s, dit levert op de vergelijking $100 = 0 + C_v$, zodat $C_v = 100$ en $v(t) = -9,81t + 100$ (m/s).

Het projectiel begint te dalen op het moment dat $v(t) = 0$:

$$v(t) = 0 \Rightarrow -9,81t + 100 = 0 \Rightarrow t = \frac{100}{9,81} = 10,1936799 \text{ sec}$$

De hoogtefunctie:

$$\begin{aligned}h &= \int v dt = \int (-9,81t + 100) dt \\ &= -4,905t^2 + 100t + C_h\end{aligned}$$

Als $t = 0$ s, dan is $h = 0$ m, zodat $C_h = 0$ en $h(t) = -4,905t^2 + 100t$ (m).

De bereikte hoogte:

$$h(10,1936799) = 509,684 \text{ m}$$

Opgave 5

We berekenen achtereenvolgens de snelheid $v(t)$, de remtijd, de afstandsfunctie $s(t)$ en uiteindelijk de remweg.

$$\begin{aligned}v(t) &= \int a(t) dt \\ &= -\int 4\sqrt{t} dt \\ &= -4 \int t^{\frac{1}{2}} dt \\ &= -\frac{8}{3}t\sqrt{t} + C_v\end{aligned}$$

Als $t = 0$ s, dan is $v = \frac{120}{3,6}$ m/s, dit levert op de vergelijking $\frac{120}{3,6} = 0 + C_v$, zodat $C_v = \frac{120}{3,6}$ en

$$v(t) = -\frac{8}{3}t\sqrt{t} + \frac{120}{3,6} \text{ (m/s)}.$$

Vervolgens lossen we op: $v(t) = 0$.

$$v(t) = 0 \Rightarrow -\frac{8}{3}t\sqrt{t} + \frac{120}{3,6} = 0 \Rightarrow t^{\frac{3}{2}} = \frac{120 \cdot 3}{3,6 \cdot 8} \Rightarrow t = \left(\frac{120 \cdot 3}{3,6 \cdot 8}\right)^{\frac{2}{3}} = 5,3860867 \text{ sec}$$

De afstandsfunctie:

$$\begin{aligned}s &= \int v dt = \int \left(-\frac{8}{3}t\sqrt{t} + \frac{120}{3,6}\right) dt \\ &= -\frac{8}{3} \int t^{\frac{3}{2}} dt + \frac{120}{3,6} \int 1 dt \\ &= -\frac{16}{15}t^{\frac{5}{2}}\sqrt{t} + \frac{120}{3,6}t + C_s\end{aligned}$$



Als $t = 0$ s, dan is $s = 0$ m, zodat $C_s = 0$ en $s(t) = -\frac{16}{15}t^2\sqrt{t} + \frac{120}{3,6}t$ (m).

De remweg:

$$s(5,3860867) = 107,722 \text{ m}$$

6.2 Bepaalde integralen

Opgave 1

a
$$\int_0^{\frac{1}{2}\pi} \sin t \, dt = [-\cos t]_0^{\frac{1}{2}\pi} = -\cos\left(\frac{1}{2}\pi\right) + \cos 0 = 1$$

b
$$\int_1^3 \frac{5}{x} \, dx = [5 \ln |x|]_1^3 = 5 \ln 3$$

c
$$\int_0^{\frac{1}{4}\pi} \frac{3}{\cos^2 x} \, dx = [3 \tan x]_0^{\frac{1}{4}\pi} = 3 \tan\left(\frac{1}{4}\pi\right) - 3 \tan 0 = 3$$

d
$$\int_1^4 \frac{1}{t\sqrt{t}} \, dt = \int_1^4 t^{-\frac{3}{2}} \, dt = \left[-2t^{-\frac{1}{2}}\right]_1^4 = -\frac{2}{\sqrt{4}} + \frac{2}{\sqrt{1}} = -1 + 2 = 1$$

e
$$\int_0^4 \frac{2}{x^2 + 16} \, dx = \frac{2}{4} \left[\arctan\left(\frac{x}{4}\right) \right]_0^4 = \frac{1}{2} (\arctan 1 - \arctan 0) = \frac{1}{8} \pi$$

f
$$\int_0^{\frac{1}{4}\pi} \cos t \, dt = [\sin t]_0^{\frac{1}{4}\pi} = \sin\left(\frac{1}{4}\pi\right) - \sin 0 = \frac{1}{2} \sqrt{2}$$

g
$$\int_0^1 (p^2 - p^3) \, dp = \left[\frac{1}{3} p^3 - \frac{1}{4} p^4 \right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

h

$$\begin{aligned} \int_1^9 \left(\frac{2}{\sqrt{x}} + 3\sqrt{x} \right) dx &= \int_1^9 \left(2x^{-\frac{1}{2}} + 3x^{\frac{1}{2}} \right) dx \\ &= \left[4x^{\frac{1}{2}} + 2x^{\frac{3}{2}} \right]_1^9 \\ &= 4 \cdot 3 + 2 \cdot 9 \cdot 3 - (4 + 2) = 60 \end{aligned}$$

i

$$\begin{aligned} \int_0^2 (3 \cdot e^t - 4^t + 7) \, dt &= \left[3 \cdot e^t - \frac{4^t}{\ln 4} + 7t \right]_0^2 \\ &= 3 \cdot e^2 - \frac{16}{\ln 4} + 14 - \left(3 - \frac{1}{\ln 4} \right) = 3 \cdot e^2 - \frac{15}{\ln 4} + 11 \end{aligned}$$



j

$$\begin{aligned}\int_0^5 (5 + x^2 \sqrt{x}) dx &= \int_0^5 \left(5 + x^{\frac{5}{2}} \right) dx \\ &= \left[5x + \frac{2}{7} x^{\frac{7}{2}} \right]_0^5 \\ &= 25 + \frac{2}{7} \cdot 125 \sqrt{5} = 25 + \frac{250}{7} \sqrt{5}\end{aligned}$$

k

$$\begin{aligned}\int_{-1}^1 (3^x - x^3) dx &= \left[\frac{3^x}{\ln 3} - \frac{1}{4} x^4 \right]_{-1}^1 \\ &= \frac{3}{\ln 3} - \frac{1}{4} - \left(\frac{1}{3 \ln 3} - \frac{1}{4} \right) = \frac{8}{3 \ln 3}\end{aligned}$$

l

$$\begin{aligned}\int_1^4 \frac{3-2s}{s\sqrt{s}} ds &= \int_1^4 s^{-\frac{3}{2}} (3-2s) ds \\ &= \int_1^4 \left(3 \cdot s^{-\frac{3}{2}} - 2 \cdot s^{-\frac{1}{2}} \right) ds = \left[-\frac{6}{\sqrt{s}} - 4\sqrt{s} \right]_1^4 = -\frac{6}{2} - 4 \cdot 2 - (-6 - 4) = -1\end{aligned}$$

Opgave 2

a $O = \int_0^3 3^x dx = \left[\frac{3^x}{\ln 3} \right]_0^3 = \frac{27}{\ln 3} - \frac{1}{\ln 3} = \frac{26}{\ln 3}$

b $O = \int_{-1}^4 3^x dx = \left[\frac{3^x}{\ln 3} \right]_{-1}^4 = \frac{81}{\ln 3} - \frac{1}{3 \ln 3} = \frac{242}{3 \ln 3}$

6.3 Integralen door substitutie

Opgave 1

a Stel $p = 5x - 3$, hieruit volgt $\frac{dp}{dx} = 5$, zodat $dx = \frac{1}{5} dp$

Er volgt:

$$\begin{aligned}\int (5x-3)^8 dx &= \frac{1}{5} \int p^8 dp \\ &= \frac{1}{45} p^{10} + C = \frac{1}{45} (5x-3)^{10} + C\end{aligned}$$

Of:

$$\begin{aligned}\int (5x-3)^8 dx &= \frac{1}{5} \int (5x-3)^8 (5x-8)' dx \\ &= \frac{1}{5} \int (5x-3)^8 d(5x-8) \\ &= \frac{1}{45} (5x-3)^{10} + C\end{aligned}$$



Toegepaste Wiskunde deel 1

b

$$\begin{aligned}\int 5 \cdot e^{8x} dx &= \frac{5}{8} \int e^{8x} \cdot (8x)' dx \\ &= \frac{5}{8} \int e^{8x} d(8x) = \frac{5}{8} e^{8x} + C\end{aligned}$$

c

$$\begin{aligned}\int x^3 \sqrt{2+5 \cdot x^4} dx &= \frac{1}{20} \int (2+5 \cdot x^4)^{\frac{1}{2}} (2+5 \cdot x^4)' dx \\ &= \frac{1}{20} \int (2+5 \cdot x^4)^{\frac{1}{2}} d(2+5 \cdot x^4) \\ &= \frac{1}{30} (2+5 \cdot x^4) \sqrt{2+5 \cdot x^4} + C\end{aligned}$$

d

$$\begin{aligned}\int \frac{x}{(6-3x^2)^5} dx &= -\frac{1}{6} \int p^{-5} dp \\ &= -\frac{1}{6} \frac{p^{-4}}{-4} + C = \frac{1}{24(6-3x^2)^4} + C\end{aligned}$$

e

Stel $p = 1 + 8t^3$, hieruit volgt $\frac{dp}{dt} = 24t^2$, zodat $t^2 dt = \frac{1}{24} dp$

Grenzen: $t = 0 \rightarrow p = 1$ en $t = 1 \rightarrow p = 9$.

Uitvoeren van de substitutie levert op:

$$\begin{aligned}\int_0^1 5t^2 \sqrt{1+8t^3} dt &= \frac{5}{24} \int_1^9 p^{\frac{1}{2}} dp \\ &= \frac{5}{24} \left[\frac{2}{3} p^{\frac{3}{2}} \right]_1^9 = \frac{5}{36} [p\sqrt{p}]_1^9 \\ &= \frac{5}{36} (9 \cdot 3 - 1) = \frac{65}{18}\end{aligned}$$

f

$$\begin{aligned}\int \frac{x^4}{1-x^5} dx &= -\frac{1}{5} \int \frac{(1-x^5)'}{1-x^5} dx \\ &= -\frac{1}{5} \int \frac{1}{1-x^5} d(1-x^5) \\ &= -\frac{1}{5} \ln|1-x^5| + C\end{aligned}$$

g

Stel $t = x^3 - 5$, hieruit volgt $\frac{dt}{dx} = 3x^2$, zodat $x^2 dx = \frac{1}{3} dt$

Grenzen: $x = 1 \rightarrow t = -4$ en $x = 3 \rightarrow t = 22$.

Uitvoeren van de substitutie levert op:



Toegepaste Wiskunde deel 1

$$\begin{aligned}\int_1^3 2 \cdot x^2 \cdot e^{x^3-5} dx &= \frac{2}{3} \int_{-4}^{22} e^t dt \\ &= \frac{2}{3} [e^t]_{-4}^{22} = \frac{2}{3} (e^{22} - e^{-4})\end{aligned}$$

h

$$\begin{aligned}\int_0^{\frac{1}{3}\pi} \cos(x) \sin^3 x dx &= \int_{x=0}^{\frac{1}{3}\pi} \sin^3 x d(\sin x) \\ &= \left[\frac{1}{4} \sin^4 x \right]_0^{\frac{1}{3}\pi} = \frac{1}{4} \sin^4 \left(\frac{1}{3}\pi \right) - 0 \\ &= \frac{1}{4} \left(\frac{1}{2} \sqrt{3} \right)^4 = \frac{1}{4} \cdot \frac{1}{16} \cdot 3^2 = \frac{9}{64}\end{aligned}$$

i

Stel $t = 5 - 2p^4$, hieruit volgt $\frac{dt}{dp} = -8p^3 \Rightarrow p^3 dp = -\frac{1}{8} dt$.

Uitvoeren van de substitutie levert op:

$$\int p^3 \cdot 3^{5-2p^4} dp = -\frac{1}{8} \int 3^t dt = -\frac{1}{8} \frac{3^t}{\ln 3} + C = -\frac{3^{5-2p^4}}{8 \ln 3} + C$$

j

$$\int \frac{1}{16+9x^2} dx = \int \frac{1}{16+(3x)^2} dx$$

Stel $p = 3x \Rightarrow \frac{dp}{dx} = 3 \Rightarrow dx = \frac{1}{3} dp$

Uitvoeren van de substitutie levert op:

$$\begin{aligned}\int \frac{1}{16+9x^2} dx &= \int \frac{1}{16+(3x)^2} dx \\ &= \frac{1}{3} \int \frac{1}{16+p^2} dp = \frac{1}{12} \arctan \left(\frac{p}{4} \right) + C \\ &= \frac{1}{12} \arctan \left(\frac{3x}{4} \right) + C\end{aligned}$$

Opgave 2

a

$$\begin{aligned}\int_0^4 \frac{4}{x^2 - 6x + 18} dx &= 4 \int_0^4 \frac{1}{(x-3)^2 + 9} dx \\ &= 4 \int_{x=0}^4 \frac{1}{(x-3)^2 + 9} d(x-3) = \frac{4}{3} \left[\arctan \left(\frac{x-3}{4} \right) \right]_0^4 \\ &= \frac{4}{3} \left(\arctan \frac{1}{4} - \arctan \left(-\frac{3}{4} \right) \right) = \frac{4}{3} \left(\arctan \frac{1}{4} + \arctan \left(\frac{3}{4} \right) \right)\end{aligned}$$

b

Stel $p = 7 - 8t^4$, hieruit volgt $\frac{dp}{dt} = -32t^3 \Rightarrow t^3 dt = -\frac{1}{32} dp$.

Uitvoeren van de substitutie levert op:



$$\begin{aligned}\int \frac{4t^3}{(7-8t^4)^5} dt &= -\frac{1}{8} \int p^{-5} dp \\ &= -\frac{1}{8} \frac{p^{-4}}{-4} + C \\ &= \frac{1}{32(7-8t^4)^4} + C\end{aligned}$$

c

$$\begin{aligned}\int_e^5 \frac{1}{x \ln^4 x} dx &= \int_e^5 (\ln x)^{-4} \frac{1}{x} dx = \int_{x=e}^5 (\ln x)^{-4} (\ln x)' dx = \int_e^5 (\ln x)^{-4} d(\ln x) \\ &= \left[-\frac{1}{3(\ln x)^3} \right]_e^5 = -\frac{1}{3(\ln 5)^3} + \frac{1}{3}\end{aligned}$$

d

$$\begin{aligned}\int_{\frac{1}{2}\pi}^{\pi} \cos^3 t dt &= \int_{\frac{1}{2}\pi}^{\pi} \cos^2 t \cdot \cos t dt \\ &= \int_{\frac{1}{2}\pi}^{\pi} (1 - \sin^2 t) d \sin t \\ &= \left[\sin t - \frac{1}{3} \sin^3 t \right]_{\frac{1}{2}\pi}^{\pi} = 0 - \left(1 - \frac{1}{3}\right) = -\frac{2}{3}\end{aligned}$$

e

$$\begin{aligned}\int \cos(3x) \cdot \sin(9x) dx &= \frac{1}{2} \int (\sin(12x) + \sin(6x)) dx \\ &= \frac{1}{2} \int \sin(12x) dx + \frac{1}{2} \int \sin(6x) dx \\ &= \frac{1}{24} \int \sin(12x) d(12x) + \frac{1}{12} \int \sin(6x) d(6x) \\ &= -\frac{1}{24} \cos(12x) - \frac{1}{12} \cos(6x) + C\end{aligned}$$

f Stel $p = 5 - 3x$. Uit $p = 5 - 3x$ volgt $x = \frac{1}{3}(5 - p)$ en $dx = -\frac{1}{3} dp$.

Uitvoeren van de substitutie levert op:

$$\begin{aligned}\int x\sqrt{5-3x} dx &= -\frac{1}{9} \int (5-p)\sqrt{p} dp \\ &= -\frac{1}{9} \int \left(5p^{\frac{1}{2}} - p^{\frac{3}{2}} \right) dp = -\frac{1}{9} \left(5 \cdot \frac{2}{3} p^{\frac{3}{2}} - \frac{2}{5} p^{\frac{5}{2}} \right) + C \\ &= -\frac{10}{27} (5-3x)\sqrt{5-3x} + \frac{2}{45} (5-3x)^2 \sqrt{5-3x} + C\end{aligned}$$



g

$$\begin{aligned}\int_0^{\frac{1}{4}\pi} \sin^3 x \cos^2 x \, dx &= \int_0^{\frac{1}{4}\pi} \sin^2 x \cos^2 x \sin x \, dx \\ &= -\int_0^{\frac{1}{4}\pi} (1 - \cos^2 x) \cos^2 x \, d(\cos x) \\ &= -\int_0^{\frac{1}{4}\pi} (\cos^2 x - \cos^4 x) \, d(\cos x) \\ &= -\left[\frac{1}{3} \cos^3 x - \frac{1}{5} \cos^5 x \right]_0^{\frac{1}{4}\pi} \\ &= -\left(\frac{1}{3} \left(\frac{1}{\sqrt{2}} \right)^3 - \frac{1}{5} \left(\frac{1}{\sqrt{2}} \right)^5 - \left(\frac{1}{3} - \frac{1}{5} \right) \right) \\ &= -\frac{1}{6\sqrt{2}} + \frac{1}{20\sqrt{2}} + \frac{2}{15} = -\frac{7}{120} \sqrt{2} + \frac{2}{15} = \frac{16 - 7\sqrt{2}}{120}\end{aligned}$$

h Stel $p = 1 + 4v$, hieruit volgt $\frac{dp}{dv} = 4$, zodat $dv = \frac{1}{4} dp$.

Uit $p = 1 + 4v$ volgt $v = \frac{1}{4}(p - 1)$.

Grenzen: $v = 0 \rightarrow p = 1$ en $v = 2 \rightarrow p = 9$.

Uitvoeren van de substitutie levert op:

$$\begin{aligned}\int_0^2 \frac{3v}{\sqrt{1+4v}} \, dv &= \frac{3}{16} \int_1^9 \frac{p-1}{\sqrt{p}} \, dp \\ &= \frac{3}{16} \int_1^9 \left(p^{\frac{1}{2}} - p^{\frac{3}{2}} \right) \, dp \\ &= \frac{3}{16} \left[\frac{2}{3} p\sqrt{p} - 2\sqrt{p} \right]_1^9 = \frac{3}{16} \left(\frac{2}{3} \cdot 9 \cdot 3 - 2 \cdot 3 \right) - \frac{3}{16} \left(\frac{2}{3} - 2 \right) = \\ &= \frac{3}{16} \cdot 12 - \frac{3}{16} \left(-\frac{4}{3} \right) = \frac{15}{6} = \frac{5}{2}\end{aligned}$$



Toegepaste Wiskunde deel 1

$$i \quad \int x^5 \sqrt{1+x^2} dx = \int x^4 \cdot x \sqrt{1+x^2} dx$$

Stel $p = 1+x^2$, hieruit volgt $\frac{dp}{dx} = 2x \Rightarrow x dx = \frac{1}{2} dp$.

Uit $p = 1+x^2$ volgt $x^2 = p-1$.

Uitvoeren van de substitutie levert op:

$$\begin{aligned} \int x^5 \sqrt{1+x^2} dx &= \int x^4 \cdot x \sqrt{1+x^2} dx \\ &= \frac{1}{2} \int (p-1)^2 \sqrt{p} dp = \frac{1}{2} \int (p^2 - 2p + 1) \sqrt{p} dp \\ &= \frac{1}{2} \int \left(p^{\frac{5}{2}} - 2p^{\frac{3}{2}} + p^{\frac{1}{2}} \right) dp \\ &= \frac{1}{2} \left(\frac{2}{7} p^{\frac{7}{2}} - 2 \cdot \frac{2}{5} p^{\frac{5}{2}} + \frac{2}{3} p^{\frac{3}{2}} \right) + C \\ &= \frac{1}{7} (1+x^2)^3 \sqrt{1+x^2} - \frac{2}{5} (1+x^2)^2 \sqrt{1+x^2} + \frac{1}{3} (1+x^2) \sqrt{1+x^2} + C \end{aligned}$$

$$j \quad \text{Gebruik } \sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \text{ zodat } \sin^2(y^5) = \frac{1}{2}(1 - \cos(2y^5))$$

Er volgt:

$$\begin{aligned} \int y^4 \sin^2(y^5) dy &= \frac{1}{2} \int y^4 (1 - \cos(2y^5)) dy \\ &= \frac{1}{2} \int y^4 dy - \frac{1}{2} \int y^4 \cos(2y^5) dy \\ &= \frac{1}{2} \int y^4 dy - \frac{1}{2} \cdot \frac{1}{10} \int \cos(2y^5) d(2y^5) \\ &= \frac{1}{10} y^5 - \frac{1}{20} \sin(2y^5) + C \end{aligned}$$

6.4 Partiële integratie

Opgave 1

a

$$\begin{aligned} \int x^5 \ln x dx &= \int x^5 \ln x d\left(\frac{1}{6} x^6\right) \\ &= \frac{1}{6} x^6 \ln x - \frac{1}{6} \int x^6 d(\ln x) \\ &= \frac{1}{6} x^6 \ln x - \frac{1}{6} \int x^6 \cdot \frac{1}{x} dx \\ &= \frac{1}{6} x^6 \ln x - \frac{1}{6} \int x^5 dx \\ &= \frac{1}{6} x^6 \ln x - \frac{1}{36} x^6 + C \end{aligned}$$



Toegepaste Wiskunde deel 1

- b We moeten de factor e^{3x} achter het d-teken brengen. Om e^{3x} achter het d-teken te kunnen brengen, bepalen we eerst $\int e^{3x} dx$:

$$\int e^{3x} dx = \frac{1}{3} \int e^{3x} d(3x) = \frac{1}{3} e^{3x} + C, \text{ zodat } e^{3x} dx = d\left(\frac{1}{3} e^{3x}\right).$$

Er volgt

$$\begin{aligned} \int_0^1 x \cdot e^{3x} dx &= \int_{x=0}^1 x d\left(\frac{1}{3} e^{3x}\right) \\ &= \frac{1}{3} \left[x \cdot e^{3x} \right]_0^1 - \frac{1}{3} \int_0^1 e^{3x} dx \\ &= \frac{1}{3} e^3 - \frac{1}{9} \left[e^{3x} \right]_0^1 \\ &= \frac{1}{3} e^3 - \frac{1}{9} (e^3 - 1) = \frac{2}{9} e^3 + \frac{1}{9} \end{aligned}$$

c

$$\begin{aligned} \int t^2 \cdot 4^t dt &= \int t^2 d\left(\frac{4^t}{\ln 4}\right) \\ &= \frac{4^t \cdot t^2}{\ln 4} - \frac{1}{\ln 4} \int 4^t dt^2 \\ &= \frac{4^t \cdot t^2}{\ln 4} - \frac{2}{\ln 4} \int 4^t \cdot t dt \\ &= \frac{4^t \cdot t^2}{\ln 4} - \frac{2}{\ln 4} \int t d\left(\frac{4^t}{\ln 4}\right) \\ &= \frac{4^t \cdot t^2}{\ln 4} - \frac{2}{\ln 4} \left(\frac{4^t \cdot t}{\ln 4} - \frac{1}{\ln 4} \int 4^t dt \right) \\ &= \frac{4^t \cdot t^2}{\ln 4} - \frac{2t \cdot 4^t}{(\ln 4)^2} + \frac{2 \cdot 4^t}{(\ln 4)^3} + C \end{aligned}$$

- d Gebruik $\sin(2x) dx = d\left(-\frac{1}{2} \cos(2x)\right)$

Er volgt:

$$\begin{aligned} \int_0^\pi x \sin(2x) dx &= \int_{x=0}^\pi x d\left(-\frac{1}{2} \cos(2x)\right) \\ &= \left[-\frac{1}{2} x \cos(2x) \right]_0^\pi + \frac{1}{2} \int_0^\pi \cos(2x) dx \\ &= -\frac{1}{2} \pi \cos(2\pi) + 0 + \frac{1}{4} \left[\sin(2x) \right]_0^\pi \\ &= -\frac{1}{2} \pi + \frac{1}{4} (\sin(2\pi) - \sin 0) = -\frac{1}{2} \pi \end{aligned}$$



Toegepaste Wiskunde deel 1

e We voeren direct partiële integratie uit volgens de formule

$$\int f(x) g'(x) dx = f(x) g(x) - \int g(x) f'(x) dx,$$

waarbij $f(x) = 5 \cdot \arctan 2x$ en $g(x) = x$:

$$\begin{aligned} \int 5 \arctan(2x) dx &= 5x \arctan(2x) - 5 \int x d(\arctan(2x)) \quad \left(d(\arctan 2x) = \frac{2}{1+4x^2} dx \right) \\ &= 5x \arctan(2x) - 10 \int \frac{x}{1+4x^2} dx \\ &= 5x \arctan(2x) - \frac{10}{8} \int \frac{(1+4x^2)'}{1+4x^2} dx \\ &= 5x \arctan(2x) - \frac{5}{4} \int \frac{1}{1+4x^2} d(1+4x^2) \\ &= 5x \arctan(2x) - \frac{5}{4} \ln(1+4x^2) + C \end{aligned}$$

f

$$\begin{aligned} \int_1^e \sqrt{x} \ln(x) dx &= \int_1^e \ln(x) d\left(\frac{2}{3} x^{\frac{3}{2}}\right) \\ &= \left[\frac{2}{3} x \sqrt{x} \ln(x) \right]_1^e - \frac{2}{3} \int_1^e x \sqrt{x} d \ln x \\ &= \frac{2}{3} e \sqrt{e} - \frac{2}{3} \int_1^e \sqrt{x} dx \\ &= \frac{2}{3} e \sqrt{e} - \frac{2}{3} \left[\frac{2}{3} x \sqrt{x} \right]_1^e \\ &= \frac{2}{3} e \sqrt{e} - \frac{4}{9} (e \sqrt{e} - 1) = \frac{2}{9} e \sqrt{e} + \frac{4}{9} \end{aligned}$$

g

$$\begin{aligned} \int 9t^2 \cdot e^{3t} dt &= 9 \int t^2 d\left(\frac{1}{3} e^{3t}\right) \\ &= 3t^2 \cdot e^{3t} - 3 \int e^{3t} d(t^2) \\ &= 3t^2 \cdot e^{3t} - 6 \int t \cdot e^{3t} dt \\ &= 3t^2 \cdot e^{3t} - 6 \int t d\left(\frac{1}{3} e^{3t}\right) \\ &= 3t^2 \cdot e^{3t} - 2t \cdot e^{3t} + 2 \int e^{3t} dt \\ &= 3t^2 \cdot e^{3t} - 2t \cdot e^{3t} + \frac{2}{3} e^{3t} + C \end{aligned}$$



h

$$\begin{aligned}\int (5x-4)\cos(3x)dx &= \int (5x-4)d\left(\frac{1}{3}\sin(3x)\right) \\ &= \frac{1}{3}(5x-4)\sin(3x) - \frac{1}{3}\int \sin(3x)d(5x-4) \\ &= \frac{1}{3}(5x-4)\sin(3x) - \frac{5}{3}\int \sin(3x)dx \\ &= \frac{1}{3}(5x-4)\sin(3x) - \frac{5}{9}\int \sin(3x)d(3x) \\ &= \frac{1}{3}(5x-4)\sin(3x) + \frac{5}{9}\cos(3x) + C\end{aligned}$$

6.5 Integratie door middel van breuksplitsing

Opgave 1

a De noemer is te ontbinden als $x^2 - 8x - 20 = (x+2)(x-10)$.

Breuksplitsen leidt tot: $\frac{4x}{(x+2)(x-10)} = \frac{A}{x-10} + \frac{B}{x+2}$. Hieruit volgt:

$$4x = A(x+2) + B(x-10)$$

$$\text{Kies } x=10: 40 = 12A \Rightarrow A = \frac{10}{3}$$

$$\text{Kies } x=-2: -8 = -12B \Rightarrow B = \frac{2}{3}$$

Er volgt:

$$\begin{aligned}\int \frac{4x}{x^2 - 8x - 20} dx &= \frac{10}{3} \int \frac{1}{x-10} dx + \frac{2}{3} \int \frac{1}{x+2} dx \\ &= \frac{10}{3} \int \frac{1}{x-10} d(x-10) + \frac{2}{3} \int \frac{1}{x+2} d(x+2) \\ &= \frac{10}{3} \ln|x-10| + \frac{2}{3} \ln|x+2| + C\end{aligned}$$

b De ontbinding van de noemer: $x^2 - 16 = (x-4)(x+4)$.

De breuksplitsen is: $\frac{3}{(x-4)(x+4)} = \frac{A}{x-4} + \frac{B}{x+4}$. Hieruit volgt:

$$3 = A(x+4) + B(x-4)$$

$$\text{Kies } x=4: 3 = 8A \Rightarrow A = \frac{3}{8}$$

$$\text{Kies } x=-4: 3 = -8B \Rightarrow B = -\frac{3}{8}$$

Er volgt:



Toegepaste Wiskunde deel 1

$$\begin{aligned}\int \frac{3}{x^2-16} dx &= \frac{3}{8} \int \frac{1}{x-4} dx - \frac{3}{8} \int \frac{1}{x+4} dx \\ &= \frac{3}{8} \int \frac{1}{x-4} d(x-4) - \frac{3}{8} \int \frac{1}{x+4} d(x+4) \\ &= \frac{3}{8} \ln|x-4| - \frac{3}{8} \ln|x+4| + C \\ &= \frac{3}{8} \ln \left| \frac{x-4}{x+4} \right| + C\end{aligned}$$

c De noemer is te ontbinden als $p^3 - 4p^2 + 4p = p(p-2)^2$.

Breuksplitsen leidt tot: $\frac{p+4}{p^3-4p^2+4p} = \frac{A}{p} + \frac{B}{p-2} + \frac{C}{(p-2)^2}$. Hieruit volgt:

$$p+4 = A(p-2)^2 + B \cdot p(p-2) + C \cdot p$$

Kies $p=0$: $4=4A \Rightarrow A=1$

Kies $p=2$: $6=2C \Rightarrow C=3$

Coëfficiënt van p^2 : $0=A+B \Rightarrow B=-1$.

Er volgt:

$$\begin{aligned}\int \frac{p+4}{p^3-4p^2+4p} dp &= \int \frac{1}{p} dp - \int \frac{1}{p-2} dp + 3 \int \frac{1}{(p-2)^2} dp \\ &= \int \frac{1}{p} dp - \int \frac{1}{p-2} d(p-2) + 3 \int (p-2)^{-2} d(p-2) \\ &= \ln|p| - \ln|p-2| - \frac{3}{p-2} + C\end{aligned}$$

d Breuksplitsen leidt tot: $\frac{x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$. Hieruit volgt:

$$x = A(x-3) + B$$

Kies $x=3$: $3=B \Rightarrow B=3$

Coëfficiënt van x : $1=A \Rightarrow A=1$.

Er volgt:

$$\begin{aligned}\int_0^2 \frac{x}{x^2-6x+9} dx &= \int_0^2 \frac{1}{x-3} dx + 3 \int_0^2 \frac{1}{(x-3)^2} dx \\ &= \int_0^2 \frac{1}{x-3} d(x-3) + 3 \int_0^2 (x-3)^{-2} d(x-3) \\ &= [\ln|x-3|]_0^2 - 3 \left[\frac{1}{x-3} \right]_0^2 \\ &= \ln|-1| - \ln|-3| - 3 \left(\frac{1}{-1} - \frac{1}{-3} \right) \\ &= -\ln(3) + 2\end{aligned}$$



Toegepaste Wiskunde deel 1

e De ontbinding van de noemer: $t^3 - 6t^2 = t^2(t-6)$.

De breuksplitsen:

$$\frac{12}{t^2(t-6)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-6}.$$

Hieruit volgt:

$$12 = At(t-6) + B(t-6) + Ct^2$$

$$\text{Kies } t = 0: 12 = -6B \Rightarrow B = -2$$

$$\text{Kies } t = 6: 12 = 36C \Rightarrow C = \frac{1}{3}$$

$$\text{Coëfficiënt van } t^2: 0 = A + C \Rightarrow A = -\frac{1}{3}.$$

Er volgt:

$$\begin{aligned} \int \frac{12}{t^3 - 6t^2} dt &= -\frac{1}{3} \int \frac{1}{t} dt - 2 \int \frac{1}{t^2} dt + \frac{1}{3} \int \frac{1}{t-6} dt \\ &= -\frac{1}{3} \ln|t| + \frac{2}{t} + \frac{1}{3} \ln|t-6| + C \end{aligned}$$

f

$$\begin{aligned} \int \frac{4x+5}{x^2+6x+13} dx &= 2 \int \frac{2x+6}{x^2+6x+13} dx - 7 \int \frac{1}{x^2+6x+13} dx \\ &= 2 \int \frac{1}{x^2+6x+13} d(x^2+6x+13) - 7 \int \frac{1}{(x+3)^2+4} d(x+3) \\ &= 2 \ln(x^2+6x+13) - \frac{7}{2} \arctan\left(\frac{x+3}{2}\right) + C \end{aligned}$$

g

$$\begin{aligned} \int_1^2 \frac{x}{x^2-8x+20} dx &= \frac{1}{2} \int_1^2 \frac{2x-8}{x^2-8x+20} dx + 4 \int_1^2 \frac{1}{x^2-8x+20} dx \\ &= \frac{1}{2} \int_1^2 \frac{1}{x^2-8x+20} d(x^2-8x+20) + 4 \int_1^2 \frac{1}{(x-4)^2+4} d(x-4) \\ &= \frac{1}{2} \left[\ln|x^2-8x+20| \right]_1^2 + 2 \left[\arctan\left(\frac{x-4}{2}\right) \right]_1^2 \\ &= \frac{1}{2} (\ln 8 - \ln 13) + 2 \left(\arctan(-1) - \arctan\left(-\frac{3}{2}\right) \right) \\ &= \frac{1}{2} \ln\left(\frac{8}{13}\right) - \frac{1}{2} \pi + 2 \arctan\left(\frac{3}{2}\right) \end{aligned}$$



Toegepaste Wiskunde deel 1

h De ontbinding van de noemer: $x^4 - 7x^3 + 10x^2 = x^2(x-5)(x-2)$.

Breuksplitsen leidt tot: $\frac{x+10}{x^2(x-5)(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-5} + \frac{D}{x-2}$. Hieruit volgt:

$$x+10 = Ax(x-5)(x-2) + B(x-5)(x-2) + C \cdot x^2(x-2) + D \cdot x^2(x-5)$$

Kies $x=0$: $10 = 10B \Rightarrow B = 1$

Kies $x=2$: $12 = -12D \Rightarrow D = -1$

Kies $x=5$: $15 = 75C \Rightarrow C = \frac{1}{5}$

Coëfficiënt van x^3 : $0 = A + C + D \Rightarrow A = -C - D = -\frac{1}{5} + 1 = \frac{4}{5}$.

Er volgt:

$$\begin{aligned} \int \frac{x+10}{x^4-7x^3+10x^2} dx &= \frac{4}{5} \int \frac{1}{x} dx + \int \frac{1}{x^2} dx + \frac{1}{5} \int \frac{1}{x-5} dx - \int \frac{1}{x-2} dx \\ &= \frac{4}{5} \ln|x| - \frac{1}{x} + \frac{1}{5} \ln|x-5| - \ln|x-2| + C \end{aligned}$$