Hoofdstuk 8 Rijen en reeksen

8.7 Herhalingsopgaven

1a
$a_0 = a = 3$, $a_1 = a + v = 3 + 1 = 4$, $a_2 = 3 + 2 = 5$, $s_2 = a_0 + a_1 + a_2 = 3 + 4 + 5 = 12$;

$$s_n = \tfrac{1}{2}(n+1)(a_0 + a_n) = \tfrac{1}{2}(n+1)(a + a + nv) = \tfrac{1}{2}(n+1)(6+n)$$

1b
$a_0 = a = 2$, $a_1 = a + v = 2 - 1 = 1$, $a_2 = 2 - 2 = 0$, $s_2 = a_0 + a_1 + a_2 = 2 + 1 + 0 = 3$;

$$s_n = \tfrac{1}{2}(n+1)(a_0 + a_n) = \tfrac{1}{2}(n+1)(a + a + nv) = \tfrac{1}{2}(n+1)(4-n)$$

2a
$a_0 = a = 2$, $a_1 = ar = 2 \cdot \tfrac{1}{2} = 1$, $a_2 = 2 \cdot \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2}$, $s_2 = a_0 + a_1 + a_2 = 2 + 1 + \tfrac{1}{2} = 3\tfrac{1}{2}$;

$$s_n = \tfrac{1}{2}(n+1)(a_0 + a_n) = \frac{a\left(1 - r^{n+1}\right)}{1-r} = 2\frac{1 - \left(\tfrac{1}{2}\right)^{n+1}}{1 - \tfrac{1}{2}} = 4\left(1 - \left(\tfrac{1}{2}\right)^{n+1}\right) = 4(1 - 2^{-n-1})$$

2b
$a_0 = a = 3$, $a_1 = ar = 3 \cdot \left(-\tfrac{1}{3}\right) = -1$, $a_2 = 3 \cdot \left(-\tfrac{1}{3}\right)^2 = \tfrac{1}{3}$, $s_2 = a_0 + a_1 + a_2 = 3 - 1 + \tfrac{1}{3} = 2\tfrac{1}{3}$;

$$s_n = \tfrac{1}{2}(n+1)(a_0 + a_n) = \frac{a\left(1 - r^{n+1}\right)}{1-r} = 3\frac{1 - \left(-\tfrac{1}{3}\right)^{n+1}}{1 - \left(-\tfrac{1}{3}\right)} = \tfrac{9}{4}\left(1 - \left(-\tfrac{1}{3}\right)^{n+1}\right) = \tfrac{9}{4}\left(1 - (-3)^{-n-1}\right)$$

3
$a_4 = 6 \Rightarrow a + 4v = 6 \Rightarrow a = 6 - 4v$ (1)

$a_0 + a_1 = a + a + v = 2a + v = 19$ (2)

(1) ingevuld in (2) geeft $2(6 - 4v) + v = 12 - 7v = 19 \Rightarrow v = -1$

Dus $a = 6 - 4v = 10$

De som van de eerste 25 termen betekent $n = 24$ in $s_n = \tfrac{1}{2}(n+1)(2a + nv)$

$s_{24} = \tfrac{1}{2}(24 + 1)(2 \cdot 10 - 24) = 50$

4
$a_3 = ar^3 = -\tfrac{1}{4}$ (1)

$\dfrac{a}{1-r} = 1\tfrac{1}{3} \Rightarrow a = \tfrac{4}{3}(1-r)$ (2)

(2) ingevuld in (1) geeft $\tfrac{4}{3}(1-r)r^3 = -\tfrac{1}{4}$

Deze 4^e-graadsvergelijking moeten we met de Solver van de GR of (bijvoorbeeld)
Maple oplossen en levert $r = -\tfrac{1}{2}$, hieruit volgt $a = \tfrac{4}{3} \cdot \tfrac{3}{2} = 2$.

5
$$\sum_{k=1}^{10}\left(\left(\tfrac{3}{4}\right)^{2k} + 1 - 3k\right) = \sum_{k=1}^{10}\left(\tfrac{9}{16}\right)^k + \sum_{k=1}^{10}(1 - 3k)$$

In de eerste som herkennen we een meetkundige reeks met 10 termen, $a = a_1 = \tfrac{9}{16}$ en

$r = \tfrac{9}{16}$, dus met som $\tfrac{9}{16}\dfrac{1 - \left(\tfrac{9}{16}\right)^{10}}{1 - \tfrac{9}{16}} = 1,281637...$

In de tweede som herkennen we een rekenkundige reeks met 10 termen, $a_1 = 1-3 = -2$ en $a_{10} = 1-30 = -29$; de som is dus $\frac{1}{2}$(aantal termen)(eerste term + laatste term) = $\frac{1}{2} \cdot 10(-2-29) = -155$

De totale som wordt dus (afgerond) $1,28164 - 155 = -153,71836$

6

$f(x) = \tan x$

$$f'(x) = \frac{1}{(\cos x)^2} = (\cos x)^{-2}$$

$$f''(x) = -2(\cos x)^{-3}(-\sin x) = 2\frac{\sin x}{(\cos x)^3} = 2f(x)f'(x)$$

$$a_0 = f(\tfrac{1}{4}\pi) = \tan(\tfrac{1}{4}\pi) = 1$$

$$a_1 = \frac{f'(\tfrac{1}{4}\pi)}{1} = \frac{1}{\left(\tfrac{1}{2}\sqrt{2}\right)^2} = 2$$

$$a_2 = \frac{f''(\tfrac{1}{4}\pi)}{2!} = \tfrac{1}{2} \cdot 2f(\tfrac{1}{4}\pi)f'(\tfrac{1}{4}\pi) = 2$$

De Taylorreeks in $x = \tfrac{1}{4}\pi$ begint dus met de termen $1 + 2(x-\tfrac{1}{4}\pi) + 2(x-\tfrac{1}{4}\pi)^2$

7a

Merk eerst op dat $f(x) = e^{\frac{2}{3}x}$, $f'(x) = \tfrac{2}{3}e^{\frac{2}{3}x}$, $f''(x) = \left(\tfrac{2}{3}\right)^2 e^{\frac{2}{3}x}$, ..., $f^{(k)}(x) = \left(\tfrac{2}{3}\right)^k e^{\frac{2}{3}x}$

$$a_0 = f(0) = 1$$

$$a_1 = \frac{f'(0)}{1} = \tfrac{2}{3}$$

$$a_2 = \frac{f''(0)}{2!} = \left(\tfrac{2}{3}\right)^2 \frac{1}{2!}$$

$$a_k = \frac{f^{(k)}(0)}{k!} = \left(\tfrac{2}{3}\right)^k \frac{1}{k!}$$

De Taylorreeks in $x = 0$ wordt dus

$$\sum_{k=0}^{\infty} \left(\tfrac{2}{3}\right)^k \frac{1}{k!} x^k = \sum_{k=0}^{\infty} \frac{2^k}{3^k} \frac{1}{k!} x^k$$

7b

$f(z) = z^2 e^{-z} \Rightarrow f(0) = 0$

$$f'(z) = 2ze^{-z} - z^2 e^{-z} = 2ze^{-z} - f(z) \Rightarrow f'(0) = 0 - f(0) = 0$$

$$f''(z) = 2e^{-z} - 2ze^{-z} - f'(z) = 2e^{-z} - 2ze^{-z} - 2ze^{-z} + f(z) = 2e^{-z} - 4ze^{-z} + f(z) \Rightarrow f''(0) = 2$$

$$f^{(3)}(z) = -2e^{-z} - 4\left(e^{-z} - ze^{-z}\right) + f'(z) = -6e^{-z} + 4ze^{-z} + 2ze^{-z} - f(z) = -6e^{-z} + 6ze^{-z} - f(z)$$

$$f^{(3)}(0) = -6$$

$$f^{(4)}(z) = 6e^{-z} + 6\left(e^{-z} - ze^{-z}\right) - f'(z) = 12e^{-z} - 6ze^{-z} - 2ze^{-z} + f(z) = 12e^{-z} - 8ze^{-z} + f(z)$$

$$f^{(4)}(0) = 12$$

$$f^{(5)}(z) = -12e^{-z} - 8\left(e^{-z} - ze^{-z}\right) + f'(z) = -20e^{-z} + 8ze^{-z} + 2ze^{-z} + f(z) = -20e^{-z} + 10ze^{-z} - f(z)$$

$$f^{(5)}(0) = -20$$

De regelmaat is nu zichtbaar:

$$f^{(k)}(z) = (-1)^k (k-1)k e^{-z} + (-1)^{k+1}(2k) ze^{-z} + (-1)^k f'(z) \Rightarrow f^{(k)}(0) = (k-1)k$$

$$a_0 = f(0) = 0$$

$$a_1 = \frac{f'(0)}{1} = 0$$

$$a_2 = \frac{f''(0)}{2!} = \tfrac{1}{2} \cdot 2 = 1$$

$$a_3 = \frac{f^{(3)}(0)}{3!} = \frac{-6}{6} = -1$$

$$a_4 = \frac{f^{(4)}(0)}{4!} = \frac{12}{24} = \frac{3 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{1}{2!}$$

$$a_5 = \frac{f^{(5)}(0)}{5!} = \frac{-20}{5!} = \frac{-4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{-1}{3!}$$

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{(-1)^k (k-1)k}{k!} = (-1)^k \frac{1}{(k-2)!} \quad \text{(vanaf } k = 2\text{)}$$

De Taylorreeks in $z = 0$ dus: $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{(k-2)!} z^k$

8a

$$f(x) = (1+x)^8 \Rightarrow f(0) = 1 \Rightarrow a_0 = 1$$

$$f'(x) = 8(1+x)^7 \Rightarrow f'(0) = 8 \Rightarrow a_1 = \frac{8}{1!} = 8$$

$$f''(x) = 8 \cdot 7(1+x)^6 \Rightarrow f''(0) = 56 \Rightarrow a_2 = \frac{56}{2!} = 28$$

Op soortgelijke wijze zien we $a_3 = \dfrac{8 \cdot 7 \cdot 6}{3!} = 56$, $a_4 = \dfrac{8 \cdot 7 \cdot 6 \cdot 5}{4!} = 70$,

$$a_4 = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{5!} = 56, \quad a_5 = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{6!} = 28, \quad a_6 = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{7!} = 8 \text{ en}$$

$a_8 = \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8!} = 1$. We zien dat vanaf $k = 9$ de k-de afgeleide 0 is, dus $a_k = 0$

vanaf $k = 9$.

Conclusie: $(1+x)^8 = 1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$

8b

$$f(z) = (z-1)^8 \Rightarrow f(0) = 1 \Rightarrow a_0 = 1$$

$$f'(z) = 8(z-1)^7 \Rightarrow f'(0) = -8 \Rightarrow a_1 = \frac{-8}{1!} = -8$$

$$f''(z) = 8 \cdot 7(z-1)^6 \Rightarrow f''(0) = 56 \Rightarrow a_2 = \frac{56}{2!} = 28$$

Op soortgelijke wijze zien we $a_3 = \dfrac{8 \cdot 7 \cdot 6}{3!} = -56$, $a_4 = \dfrac{8 \cdot 7 \cdot 6 \cdot 5}{4!} = 70$,

$a_4 = \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{5!} = -56$, $a_5 = \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{6!} = 28$, $a_6 = \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{7!} = -8$ en

$a_8 = \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8!} = 1$. We zien dat vanaf $k = 9$ de k-de afgeleide 0 is, dus $a_k = 0$ vanaf $k = 9$.

Conclusie: $(z-1)^8 = 1 - 8z + 28z^2 - 56z^3 + 70z^4 - 56z^5 + 28z^6 - 8z^7 + z^8$

9
```
> taylor((1-v)^3,v=0,4);
```
$$1 - 3v + 3v^2 - v^3$$

```
> taylor((1-z)^(-1/3),z=0,4);
```
$$1 + \frac{1}{3}z + \frac{2}{9}z^2 + \frac{14}{81}z^3 + O(z^4)$$

```
> taylor((1-z)^(-1/4),z=0,4);
```
$$1 + \frac{1}{4}z + \frac{5}{32}z^2 + \frac{15}{128}z^3 + O(z^4)$$

```
> taylor((16+w)^(-1/4),w=0,4);
```
$$\frac{1}{16}16^{(3/4)} - \frac{1}{1024}16^{(3/4)}w + \frac{5}{131072}16^{(3/4)}w^2 - \frac{15}{8388608}16^{(3/4)}w^3 + O(w^4)$$

```
> evalf(%);
```
$$0.5000000000 - 0.007812500000\,w + 0.0003051757813\,w^2 - 0.00001430511474\,w^3 + O(w^4)$$

Met `convert(%,polynom)` kunnen we de ordeterm desgewenst verwijderen.

10
```
> g:=x->(cos(x))^2;
```
$$g := x \,\circledR\, \cos(x)^2$$

```
> taylor((cos(x))^2,x=0,6);
```
$$1 - x^2 + \frac{1}{3}x^4 + O(x^6)$$

```
> convert(%,polynom);
```
$$1 - x^2 + \frac{1}{3}x^4$$

```
> f:=unapply(%,x);
```
$$f := x \,\circledR\, 1 - x^2 + \frac{1}{3}x^4$$

```
> f(Pi/3);
```

$$1 - \frac{1}{9} p^2 + \frac{1}{243} p^4$$

```
> %-g(Pi/3);
```

$$\frac{3}{4} - \frac{1}{9} p^2 + \frac{1}{243} p^4$$

```
> a:=evalf(%);
```

$$a := 0.0542377461$$

```
> h:=x->(sin(x))^2;
```

$$h := x \,®\, \sin(x)^2$$

```
> taylor(h(x),x=0,6);
```

$$x^2 - \frac{1}{3} x^4 + O(x^6)$$

```
> convert(%,polynom);
```

$$x^2 - \frac{1}{3} x^4$$

```
> k:=unapply(%,x);
```

$$k := x \,®\, x^2 - \frac{1}{3} x^4$$

```
> k(Pi/3);
```

$$\frac{1}{9} p^2 - \frac{1}{243} p^4$$

```
> %-h(Pi/3);
```

$$\frac{1}{9} p^2 - \frac{1}{243} p^4 - \frac{3}{4}$$

```
> b:=evalf(%);
```

$$b := -0.0542377461$$

```
> a+b;
```

$$0.$$

Aan de antwoorden met de waarden a en b zien we dat de reeksen nog niet voldoende nauwkeurigheid geven. Het verschil met de werkelijke functiewaarden is ongeveer 0,05. De nauwkeurigheid in de benadering kan alleen worden verbeterd door meer termen te nemen.
Merk op: $(\cos x)^2 + (\sin x)^2 = 1$, voor alle x, ook voor de Taylorreeksbenaderingen.